Chapter 2

Motion in Complex Space

Chapter Outline

The simple Hamilton mechanics introduced in chapter 1 can be used to solve almost all the non-relativistic and relativistic the quantum mechanical problems if we extend canonical variables to a complex domain. The complex-extended Hamilton mechanics, or complex mechanics by short, is dedicated to the development of a general theory unifying classical mechanics, quantum mechanics, and relativistic mechanics in complex space, and to the conveyance of the philosophy that what have been considered as probabilistic quantum events have a common origin from the particle's deterministic motion in complex space. We postulate that the actual scenario of dynamic motion happens in complex space and what we customarily consider as physical reality is merely the projection of the actual scenario into the real space. The proposed theory employs complex-extended Hamilton mechanics to describe and model quantum systems in such a way that all the particle-like properties can be reserved due to its classical nature and in the meanwhile, all the wave-like properties are manifested naturally via the multi-path behavior of complex trajectories. The proposed framework of complex mechanics makes use of classical concepts and tools to deal with particle's quantum behavior by the introduction of a complex Hamiltonian from which complex Hamilton equations describing particle's quantum motion are derived in a form of Newton's second law defined in complex space. Complex mechanics is then connected with quantum mechanics by showing the equivalence between the complex Hamilton-Jacobi equation and Schrödinger equation. The solutions of the complex Hamilton equations give us the complex trajectories traced by a particle, which are found to be non-unique. It is this non-uniqueness of the complex trajectories projected into real space that produces the multi-path phenomenon and the observed wave behavior of a material particle in the real space.

2.1 Literature Survey

The necessity of analyzing quantum phenomena in terms of complex trajectories and complex potentials has long been recognized in many branches of quantum physics. It was known that the solution of the Schrödinger equation could be expressed in terms of complex classical paths out of which wave behavior can be constructed (Balian and Bloch 1974). The studies in chaotic tunneling have also revealed that only by including complex trajectories, can the tunneling effect of transition to classically inaccessible regions be fully explained (Shudo and Ikeda 1996). Especially, the complex trajectories having no connection with the real manifold, called Laputa branches, play a crucial role to

generate chaotic tunneling. As to tunneling time, a traversal time can be unambiguously defined as the time spent by a particle between given initial and final positions on a complex trajectory. (Xavier and de Aguiar, 1997). Besides tunneling problems, complex trajectory also has successful applications in scattering problems. A highly accurate approximation of quantum scattering by a hard sphere, valid for complete range of scattering angles, has been proposed using complex-valued angular momentum (Nussenzveig and Wiscombe, 1991).

A primary motivation of extending standard quantum mechanics to complex domain is the studies of Hamiltonians with complex-valued potentials, which appeared firstly in nuclear physics, and is called optical or average nuclear potentials (Bethe, 1940). Quantum analysis using complex Hamiltonian is not only a mathematical tool but also has concrete physical realization. For example, a delocalization phenomenon was found for a non-Hermite Hamiltonian containing a constant imaginary vector potential (Hatano, 1998). As the imaginary vector potential increases, all of originally localized eigenfunctions get delocalized one by one. This delocalization phenomenon caused by complex potential has a physical realization as flux-line depining in type-II superconductors (Hatano and Nelson, 1996).

Hamiltonian with complex potential has complex eigenfunctions and is non-Hermitian in general, which, at first glance, does not satisfy the Hermitian property, required by the standard quantum mechanics to ensure the reality of energy spectrum. However, the researches of complex potentials have proved the fact that Hermiticity of the Hamiltonian is not essential for a real spectrum. Replacing the Hermiticity condition by a weaker condition called PT symmetry (Bender and Boettcher, 1998), one can obtain new classes of complex Hamiltonians whose spectra are still real and positive. Now we know that what is responsible for real spectrum is pseudo-Hermiticity (Mostafazadeh, 2002), which is a further generalization of the concept of PT symmetry. During the latest several years, many pseudo-Hermitian complex potentials have been synthesized by different approaches such as supersymmetrization (Vincenzo and Alonso, 2002), group theoretic methods (Bagchi and Quesne, 2002), quasi-solvability (Khare and Mandal, 2000), and potential algebras (Levai, Cannata, and Ventura, 2001). A crucial problem confronting such complex Hamiltonians is whether they can define a physically viable quantum mechanics, since in complex domain we encounter the severe difficulty of dealing with Hilbert spaces endowed with indefinite metrics. This problem was resolved lately by the concept of CPT symmetry (Bender, Brody, and Jones, 2002) based on which quantum mechanics with probability interpretation has been extended successfully to complex domain.

From the viewpoint of classical mechanics, when we deal with particle motion subjected to the complex potentials mentioned previously, we inevitably encounter the concept of complex position and complex momentum. One of the methods of complexifying a real quantum Hamiltonian $H(q,p) \in \mathbb{R}$ is via the complexification of position and momentum (Kaushal and Parthasarathi, 2002)

$$q = q_R + iq_I \in \mathbb{C}, \quad p = p_R + ip_I \in \mathbb{C}. \tag{2.1.1}$$

where q_R, p_R, and q_I, p_I, are the real parts and the imaginary parts of the complex position q and the complex momentum p, respectively. If we release the restriction on the particle's motion in real domain and allow the particle to movie in complex space, then the particle's motion will be governed by the complex Hamiltonian

$$H(q,p) = H(q_R + iq_I, p_R + ip_I) = H_1(q_R, p_R, x_I, p_I) + iH_2(q_R, p_R, q_I, p_I) \in \mathbb{C}. \tag{2.1.2}$$

By extending the methods used in classical mechanics to complex domain, one can obtain the complex trajectories traced by the particle, along which the associated complex-motion invariants can also be identified (Kaushal and Singh, 2001). Complex solutions to classical Hamilton-Jacobi equation were also helpful in explaining the potential tunneling in which the total energy of a particle is lower than its surrounding potential, and in explaining the dynamical tunneling that connects energetically accessible but dynamically separated classical paths (Takatsuka and Ushiyama, 1995).

The relations of Schrödinger equation to particle's motion in complex space have been pointed out by several authors. The first relation comes from a recent discovery about a new interpretation of Schrödinger and Klein-Gordon equations (Baker-Jarvis and Kabos, 2003), wherein conservation of

probability is replaced with conservation of energy in complex domain. In this new interpretation, the complex energy $E = E_R + iE_I \in \mathbb{C}$ is defined in terms of a complex action $S = S_R + iS_I \in \mathbb{C}$ via the relation:

$$E = E_R + iE_I = -\frac{\partial S}{\partial t} = -\frac{\partial S_R}{\partial t} - i\frac{\partial S_I}{\partial t}\,, \tag{2.1.3}$$

where E_R is attributed to particle energy, while E_I is due to particle-wave interaction energy. The complex momentum $\boldsymbol{p} = \boldsymbol{p}_R + i\,\boldsymbol{p}_I$ is defined in terms of S as

$$\boldsymbol{p} = \boldsymbol{p}_R + i\,\boldsymbol{p}_I = \nabla S = \nabla S_R + i\nabla S_I \in \mathbb{C}^3\,, \tag{2.1.4}$$

where $\boldsymbol{p}_R \in \mathbb{R}^3$ is identified with the particle momentum and $\boldsymbol{p}_I \in \mathbb{R}^3$ is with the particle-wave interaction momentum. By making use of the above notions of complex energy and complex momentum, it was shown in (Baker-Jarvis and Kabos, 2003) that instead of viewing Schrödinger and Klein-Gordon equations as the conservation of probability, one could reveal their alternative role as the conservation of energy. This point-wise conservation is valid for each spatial point in complex space, which is different from the conventional sense of conservation defined as the assemble average in real space.

The other relation of Schrödinger equation to particle's complex motion stems from the Nelson's derivation of the Schrödinger equation from Newtonian Mechanics (Nelson, 1966). Nelson showed that an entirely Newtonian derivation of the Schrödinger equation could be given by considering Brownian motion with diffusion coefficient $\mathcal{D} = \hbar\,/\,2m$. Nelson's derivation of Schrödinger equation is in terms of two velocity components in Brown motion: current velocity $\boldsymbol{v}$ and the osmotic velocity $\boldsymbol{u}$ defined as

$$\boldsymbol{v} = \frac{1}{2}(\boldsymbol{b}_+ + \boldsymbol{b}_-), \quad \boldsymbol{u} = \frac{1}{2}(\boldsymbol{b}_+ - \boldsymbol{b}_-)\,, \tag{2.1.5}$$

where $\boldsymbol{b}_+$ and $\boldsymbol{b}_-$ are, respectively, the mean forward and mean backward velocities appeared in Brownian motion. If we express $\boldsymbol{v}$ and $\boldsymbol{u}$ in terms of two new functions S_N and R_N defined as

$$\boldsymbol{v} = \frac{\hbar}{m}\nabla S_N, \quad \boldsymbol{u} = \frac{\hbar}{m}\nabla R_N\,. \tag{2.1.6}$$

Then Nelson showed that the function

$$\Psi = e^{R_N + iS_N}\,, \tag{2.1.7}$$

satisfies the Schrödinger equation. A noticeable observation from Nelson's approach is that the wave function Ψ can be synthesized by a complex velocity with real part $\boldsymbol{v}$ and imaginary part $\boldsymbol{u}$. This result again implies that the concept of complex motion is inherent in the Schrödinger 's approach to quantum mechanics.

The most remarkable existing work on complex motion is Nottale's theory of scale relativity (Nottale, 1998). Scale relativity leads naturally to the concept of fractal space-time, which describes quantum space-time as a non-differentiable fractal continuum. Nottale's fractal hypothesis gives rise to the enlightening result that the Nelson's Brownian approach to quantum mechanics can be simply reproduced by replacing the classical time-derivative in Newtonian mechanics by a new complex covariant derivative:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{\mathcal{V}}\bullet\nabla - i\frac{\hbar}{2m}\nabla^2\,, \tag{2.1.8}$$

where $\boldsymbol{\mathcal{V}}$ is the complex velocity formed from $\boldsymbol{v}$ and $\boldsymbol{u}$ in Eq.(2.1.6)

$$\boldsymbol{\mathcal{V}} = \boldsymbol{v} - i\,\boldsymbol{u}\,. \tag{2.1.9}$$

Nottale assumes that any mechanical system can be characterized by a Lagrange function $\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$, from which a complex action S is defined:

$$S = \int_{t_1}^{t_2} \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) dt \,, \tag{2.1.10}$$

where the complex Lagrange function $\mathcal{L}$ is obtained from the classical Lagrange function $L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ by the substitution $d\,/\,dt \to \mathcal{d}\,/\,dt$:

$$\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \mathcal{L}(\boldsymbol{q}, d\boldsymbol{q}\,/\,dt, t) \to L(\boldsymbol{q}, \mathcal{d}\boldsymbol{q}\,/\,dt, t) = \mathcal{L}(\boldsymbol{q}, \boldsymbol{\mathcal{V}}, t)\,. \tag{2.1.11}$$

The principle of stationary action, $\delta S = 0$, applied to this complex action with both ends of the above integral fixed, leads to complex Euler-Lagrange equations:

$$\frac{\mathcal{d}}{dt}\frac{\partial \mathcal{L}}{\partial \mathcal{V}_i} = \frac{\partial \mathcal{L}}{\partial q_i}, \tag{2.1.12}$$

which, in turn, reduces to the complex Newtonian equation

$$m\frac{\mathcal{d}}{dt}\boldsymbol{\mathcal{V}} = -\nabla V\,. \tag{2.1.13}$$

By making use of the definition of $\mathcal{d}\,/\,dt$ in Eq.(2.1.8) and noting the relation between the complex velocity $\boldsymbol{\mathcal{V}}$ and the wave function Ψ from Eq.(2.1.6) and Eq.(2.1.7)

$$\boldsymbol{\mathcal{V}} = \boldsymbol{v} - i\,\boldsymbol{u} = \frac{\hbar}{mi}\nabla(R_N + iS_N) = \frac{\hbar}{mi}\nabla \ln \Psi\,, \tag{2.1.14}$$

it can be show that the complex Newtonian equation (2.1.13) is exactly identical to the Schrödinger 's equation

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi\,. \tag{2.1.15}$$

This inspirational result reveals the clue to the conjecture that classical mechanics and quantum mechanics may have their common root in complex space. The other consequence of Nottale's approach is the connection between complex-space formulation and multi-path formulation. The concept of fractal space-time has an origin from Feynman's multi-path idea (Feynman and Hibbs, 1965), which allows an infinity of geodesic lines between any couple of points in a fractal space-time so that the non-relativistic quantum mechanics can be reformulated as a multi-path integral over Brownian paths. Nottale's complex formulation of quantum mechanics underpins the implication that the projection of complex paths to real space naturally gives rise to the multi-path phenomenon observed in real space. That is to say, what is known as multi-path phenomenon is nothing but the projection effect from complex space to real space. We will prove this point in later sections.

The above-related researches in classical and quantum mechanics in complex domain provide us with good reason to consider seriously the possibility that actual particle motion happens in complex space, but what we have sensed and measured is only the real part of the motion, which constitutes the real physical world that we experience in daily life. Accordingly, the establishment of complex mechanics proposed here is based on such a postulate that the actual scenario of dynamic motion happens in complex space and what we customarily consider as physical reality is merely the projection of the actual scenario into the real space. Within complex domain, we will find that classical mechanics and quantum mechanics can be made compatible with each other and can be further incorporated into a unified framework - called complex mechanics here. Complex mechanics can be regarded as the complex extension of classical mechanics, hence all the particle-like (local) properties are reserved; in the meanwhile, the multi-path behavior of complex trajectories manifest all the wave-like (non-local) properties exhibiting in quantum mechanics. Therefore, under the framework of complex mechanics, we can make use of classical concepts and tools to deal with quantum behavior of particles.

2.2 Motions in Complex Space

The motion of an n degree-of-freedom dynamic system treated in complex mechanics is

described in terms of the complex phase space $(\boldsymbol{q},\boldsymbol{p})\in\mathbb{C}^n\times\mathbb{C}^n$, which is recognized here as a space with physical reality (Yang, 2007C). The first postulate of complex mechanics is that the actual scenario of dynamic motion happens in complex space and what we customarily considered as physical reality is merely the projection of the actual scenario into the real space. We first consider a particle moving within the complex space. Its position vector $\boldsymbol{q}$ is described by the complex coordinates $(q_1\ q_2\ q_3)$ with real part $\boldsymbol{q}_R$ and imaginary part $\boldsymbol{q}_I$ given as

$$\boldsymbol{q}(t)=\begin{bmatrix}q_1 & q_2 & q_3\end{bmatrix}^T=\begin{bmatrix}q_{R_1} & q_{R_2} & q_{R_3}\end{bmatrix}^T+i\begin{bmatrix}q_{I_1} & q_{I_2} & q_{I_3}\end{bmatrix}^T=\boldsymbol{q}_R(t)+i\,\boldsymbol{q}_I(t)\in\mathbb{C}^3\,, \tag{2.2.1}$$

and analogously its momentum is described by the complex vector $\boldsymbol{p}$:

$$\boldsymbol{p}(t)=\begin{bmatrix}p_1 & p_2 & p_3\end{bmatrix}^T=\begin{bmatrix}p_{R_1} & p_{R_2} & p_{R_3}\end{bmatrix}^T+i\begin{bmatrix}p_{I_1} & p_{I_2} & p_{I_3}\end{bmatrix}^T=\boldsymbol{p}_R(t)+i\,\boldsymbol{p}_I(t)\in\mathbb{C}^3\,. \tag{2.2.2}$$

In complex mechanics, the introduction of the imaginary phase space $(\boldsymbol{q}_I,\boldsymbol{p}_I)$ is not merely for the purpose of mathematical analysis; here $(\boldsymbol{q}_I,\boldsymbol{p}_I)$ possesses the same degree of physical reality as $(\boldsymbol{q}_R,\boldsymbol{p}_R)$ does. Although $(\boldsymbol{q}_I,\boldsymbol{p}_I)$ cannot be directly sensed or measured in the real world, they do exist in the complex space. In spite of the unobservability of $(\boldsymbol{q}_I,\boldsymbol{p}_I)$, their influences on $(\boldsymbol{q}_R,\boldsymbol{p}_R)$ can be definitely detected via the measurement of $(\boldsymbol{q}_R,\boldsymbol{p}_R)$. In other wards, we may treat $(\boldsymbol{q}_I,\boldsymbol{p}_I)$ as the hidden variables in complex mechanics. It is just the coupling effect between $(\boldsymbol{q}_R,\boldsymbol{p}_R)$ and $(\boldsymbol{q}_I,\boldsymbol{p}_I)$ that gives rise to the quantum phenomena we have observed in the real world, such as wave behavior of particle, spin, and tunneling effect, etc. One of the major goals of complex mechanics considered here is to demonstrate how the coupling effect between $(\boldsymbol{q}_R,\boldsymbol{p}_R)$ and $(\boldsymbol{q}_I,\boldsymbol{p}_I)$ relates to such non-classical phenomena.

If a quantum theory completely ignores the role of $(\boldsymbol{q}_I,\boldsymbol{p}_I)$ and concentrates only on the measurement analysis of $(\boldsymbol{q}_R,\boldsymbol{p}_R)$, it will unavoidably find some uncertainties or ambiguities accompanying with $(\boldsymbol{q}_R,\boldsymbol{p}_R)$. As an example of this speculation, $(\boldsymbol{q}_R,\boldsymbol{p}_R)$ can be the regarded as the hidden variables in Bohm's causal interpretation (Bohm, 1952) of quantum mechanics, without considering the role of $(\boldsymbol{q}_I,\boldsymbol{p}_I)$. Hence, Bohm's approach may remove the ambiguity caused by $(\boldsymbol{q}_R,\boldsymbol{p}_R)$ from the standard quantum mechanics but the uncertainty originated from $(\boldsymbol{q}_I,\boldsymbol{p}_I)$ is still resident in his formulation. Just to take into account this uncertainty, in a later paper (Bohm and Vigier, 1954) Bohm proposed an additional irregular fluctuating component in conjunction with his original guidance condition. Instead of considering only $(\boldsymbol{q}_R,\boldsymbol{p}_R)$ as in Bohm's approach, complex mechanics treats $(\boldsymbol{q}_R,\boldsymbol{p}_R)$ and $(\boldsymbol{q}_I,\boldsymbol{p}_I)$ simultaneously so that in complex space particle's motion can be identified deterministically and all the uncertainties coming from $(\boldsymbol{q}_R,\boldsymbol{p}_R)$ and $(\boldsymbol{q}_I,\boldsymbol{p}_I)$ disappear automatically; when we project the complex motion into real space as is done in any measurement process, uncertainties arise naturally due to the neglect of the imaginary motion $(\boldsymbol{q}_I,\boldsymbol{p}_I)$.

Having defined the complex phase space $(\boldsymbol{q},\boldsymbol{p})\in\mathbb{C}^3\times\mathbb{C}^3$, we proceed to investigate how a particle moves in this space. The second postulate of complex mechanics concerns with the formulation of complex Hamiltonian

$$H(\boldsymbol{q},\boldsymbol{p},t)=\frac{1}{2m}\,\boldsymbol{p}\bullet\boldsymbol{p}+V(\boldsymbol{q},t)+Q(\boldsymbol{q},t)\,, \tag{2.2.3}$$

where $V(\boldsymbol{q},t)$ is the applied external potential and the complex potential $Q(\boldsymbol{q},t)$ is defined as

$$Q(\boldsymbol{q},t)=\frac{1}{2m}\left(\frac{\hbar}{i}\right)\nabla\bullet\boldsymbol{p}=\frac{1}{2m}\left(\frac{\hbar}{i}\right)\left(\frac{\partial p_1(\boldsymbol{q},t)}{\partial q_1}+\frac{\partial p_2(\boldsymbol{q},t)}{\partial q_2}+\frac{\partial p_3(\boldsymbol{q},t)}{\partial q_3}\right). \tag{2.2.4}$$

This state-dependent complex potential $Q(\boldsymbol{q},t)$ is a measure of the spatial distribution of momentum $\boldsymbol{p}$, which is intrinsic, but not specified externally. In the following we will show that the complex dynamics represented by the Hamiltonian (2.2.3), with complex potential $Q(\boldsymbol{q},t)$ given by Eq.(2.2.4), is compatible with the quantum dynamics determined by the Schrödinger 's equation with external potential $V(\boldsymbol{q},t)$. Complex potential $Q(\boldsymbol{q},t)$ provides an important connection between classical mechanics and quantum mechanics. Without this term, a direct complexification of the classical system $H=\boldsymbol{p}\bullet\boldsymbol{p}/2m+V(\boldsymbol{q},t)$ by using Eq.(2.2.1) and Eq.(2.2.2) still cannot establish the equivalence with quantum mechanics.

To derive the equations of motion within the complex space, we form the complex action S as

$$S = \int_{t_1}^{t_2} \left(\boldsymbol{p} \bullet \dot{\boldsymbol{q}} - H(\boldsymbol{q}, \boldsymbol{p}, t) \right) dt \,. \tag{2.2.5}$$

The principle of stationary action $\delta S = 0$, with the variations $\delta \boldsymbol{q}$ and $\delta \boldsymbol{p}$ taken in $\mathbb{C}^3$, leads to the complex Hamilton equations:

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad q_j(t_0) = q_j^0 \in \mathbb{C}, \quad j = 1,\ 2,\ 3\,, \tag{2.2.6a}$$

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad p_j(t_0) = p_j^0 \in \mathbb{C}, \quad j = 1,\ 2,\ 3\,. \tag{2.2.6b}$$

Note that in the expression of $Q(\boldsymbol{q}, t)$ in Eq.(2.2.4), the momentum $\boldsymbol{p}$ is conceived of as an explicit function of $\boldsymbol{q}$ (in chapter 2, we will see how this is done); consequently, the variation $\delta Q(\boldsymbol{q}, t)$ is solely due to the variation of $\boldsymbol{q}$, as in the case of $V(\boldsymbol{q}, t)$. Although the form of Eqs.(2.2.6) is, in appearance, identical to the classical Hamilton equation, there are two remarkable properties different from the classical Hamilton equations:

(1) Both q_j and p_j are complex variables, instead of real variables.

(2) The Hamiltonian $H(\boldsymbol{q}, \boldsymbol{p}, t)$ has an additional term due to the intrinsic complex potential $Q(\boldsymbol{q}, t)$.

If we consider the complex action as a function of the upper limit of the integration in Eq.(2.2.5), the variation of the action S yields an expression for the complex momentum $\boldsymbol{p}$, as well as a complex Hamilton-Jacobi equation:

$$\boldsymbol{p} = \nabla S \in \mathbb{C}^3\,, \tag{2.2.7a}$$

$$\frac{\partial S}{\partial t} + H(\boldsymbol{q}, \boldsymbol{p}, t)\big|_{\boldsymbol{p} = \nabla S} = 0\,. \tag{2.2.7b}$$

Eq.(2.2.7a) can be recognized as the complex guidance condition in comparison with the real guidance condition postulated in the de Broglie-Bohm approach (Bohm, 1952). But what is different here is that the complex guidance condition (2.2.7a) is a derivation but not a postulate of complex mechanics. The derivation of Eq.(2.2.7a) from the complex Hamilton principle also legitimates the use of ∇S as complex momentum proposed in (Baker-Jarvis and Kabos, 2003).

As a matter of fact, we can show that Eq.(2.2.6) and Eq.(2.2.7) are equivalent. We will begin with Eq.(2.2.6) and show that Eq.(2.2.7) is achieved naturally. Starting with Eq.(2.21), we assume a solution for p_j in the form of

$$p_j = p_j(\boldsymbol{q}, t) = \frac{\partial \overline{S}(\boldsymbol{q}, t)}{\partial q_j}, \quad j = 1,\ 2,\ 3\,, \tag{2.2.8}$$

where $\overline{S}(\boldsymbol{q}, t)$ is a function to be determined, which is assumed to be twice continuously differentiable with respect to q_j. With this form of solution for p_j, we have the identity

$$\frac{\partial p_j}{\partial q_k} = \frac{\partial}{q_k}\frac{\partial}{q_j}\overline{S} = \frac{\partial}{q_j}\frac{\partial}{q_k}\overline{S} = \frac{\partial p_k}{\partial q_j}, \quad j,\ k = 1,\ 2,\ 3 \tag{2.2.9}$$

Next we rewrite Eq.(2.2.6b) as

$$\frac{dp_j}{dt} = \frac{\partial p_j}{\partial t} + \sum_{k=1}^{3} \frac{\partial p_j}{\partial q_k}\dot{q}_k = \frac{\partial}{\partial t}\frac{\partial \overline{S}}{\partial q_j} + \sum_{k=1}^{3} \frac{\partial p_j}{\partial q_k}\frac{\partial H}{\partial p_k} = -\frac{\partial H}{\partial q_j}\,,$$

which, in conjunction with Eq.(2.2.9) yields

$$\frac{\partial}{\partial t}\frac{\partial \overline{S}}{\partial q_j} + \sum_{k=1}^{3} \frac{\partial p_k}{\partial q_j}\frac{\partial H}{\partial p_k} + \frac{\partial H}{\partial q_j} = 0\,. \tag{2.2.10}$$

In terms of the total differentiation D / Dq_j defined as

$$\frac{D}{Dq_j} H(\boldsymbol{q}, \boldsymbol{p}(\boldsymbol{q}), t) = \frac{\partial H}{\partial q_j} + \sum_{k=1}^{3} \frac{\partial p_k}{\partial q_j}\frac{\partial H}{\partial p_k}\,,$$

Eq.(2.2.10) becomes

$$\frac{D}{Dq_j}\left(\frac{\partial \overline{S}}{\partial t}+H\right)=0, \quad j=1,\ 2,\ 3\,, \tag{2.2.11}$$

where $D\overline{S}\,/\,Dq_j=\partial\overline{S}\,/\,\partial q_j$, since $\overline{S}(\boldsymbol{q},t)$ depends on q_j only. Eq.(2.2.11) shows that $\partial\overline{S}\,/\,\partial t+H$ must be a function only of t, i.e.,

$$\frac{\partial \overline{S}}{\partial t}+H(q_1,q_2,q_3,p_1,p_2,p_3,t)\big|_{p_j=\partial\overline{S}\,/\,\partial q_j}=f(t)\,.$$

The arbitrary time function $f(t)$ can be absorbed in the unknown function $\overline{S}$, leading to

$$\frac{\partial \overline{S}}{\partial t}+H(\boldsymbol{q},\boldsymbol{p},t)\big|_{\boldsymbol{p}=\nabla\overline{S}}=0\,. \tag{2.2.12}$$

Therefore as expected, we begin with the complex Hamilton equation (2.2.6) and finally arrive at the complex Hamilton-Jacobi equation (2.2.12).

The complex Hamilton-Jacobi equation in Eq.(2.2.7b) or (2.2.12) is nothing but an alternative expression for time-dependent Schrödinger equation. Applying Eq.(2.2.3) to Eq.(2.2.7b), we obtain an explicit expression for the complex Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t}+\frac{1}{2m}\nabla S\bullet\nabla S+V(q,t)+\frac{1}{2m}\left(\frac{\hbar}{i}\right)\nabla^2 S=0\,. \tag{2.2.13}$$

Substituting the transformation

$$\Psi=e^{iS\,/\,\hbar}, \quad \text{or} \quad S=-i\hbar\ln\Psi\,, \tag{2.2.14}$$

into Eq.(2.2.13), yields an equation for the function Ψ

$$i\hbar\frac{\partial\Psi(\boldsymbol{q},t)}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\Psi(\boldsymbol{q},t)+V\Psi(\boldsymbol{q},t)\,, \tag{2.2.15}$$

which is just the standard form of Schrödinger equation. In terms of the wave function Ψ, the complex potential Q defined in Eq.(2.2.4) can be rewritten as

$$Q=\frac{1}{2m}\left(\frac{\hbar}{i}\right)\nabla\bullet\boldsymbol{p}=\frac{1}{2m}\left(\frac{\hbar}{i}\right)\nabla^2 S=-\frac{\hbar^2}{2m}\nabla^2\ln\Psi\,. \tag{2.2.16}$$

What we have shown above is the equivalence among the complex Hamilton equation (2.2.6), the complex Hamilton-Jacobi equation (2.2.7a), and the Schrödinger equation (2.2.15). Of greater significance is the implication behind this equivalence: we can now treat quantum problems completely in terms of Eq.(2.2.6) and Eq.(2.2.7), which are complex extension of classical mechanics, and which allow us to employ classical concepts and tools (with proper complex extension) to describe what Schrödinger 's equation can describe in quantum mechanics. We will show this powerfulness of complex mechanics in the following sections by solving typical quantum problems without giving up our intuition gained from classical mechanics.

The role of wave function Ψ in complex mechanics is not as critical as in quantum mechanics. In fact we can solve quantum problems by employing Eq.(2.2.6) directly without resorting to Schrödinger 's equation (2.2.15). Under the framework of complex mechanics, the necessity of Schrödinger 's equation comes from its linearity and its easier solvability than that of the complex Hamilton-Jacobi equation (2.2.13), which is a nonlinear partial differential equation. Once S or Ψ is found, particle momentum $\boldsymbol{p}$ is then determined from Eq.(2.2.7a):

$$p_j(\boldsymbol{q},t)=\frac{\hbar}{i}\frac{1}{\Psi(\boldsymbol{q},t)}\frac{\partial\Psi(\boldsymbol{q},t)}{\partial q_j}, \quad j=1,\ 2,\ 3\,. \tag{2.2.17}$$

Using this obtained $p_j(\boldsymbol{q},t)$ in Eq.(2.2.6a), we can solve for the particle's trajectory $\boldsymbol{q}(t)$:

$$\frac{dq_j}{dt}=\frac{\partial H}{\partial p_j}=\frac{p_j(\boldsymbol{q},t)}{m}, \quad q_j(t_0)=q_j^0\in\mathbb{C}, \quad j=1,\ 2,\ 3\,. \tag{2.2.18}$$

Along the above approach, we are led to the understanding that in complex mechanics, $\boldsymbol{q}(t)$ and

$\boldsymbol{p}(t)$ are of realistic existence, but not the wavefunction Ψ; the latter only provides a convenient mathematical medium for solving the former. Even in the description of particle's wave behavior discussed in chapter 6, the main role is also played by $\boldsymbol{q}(t)$ and $\boldsymbol{p}(t)$; Ψ only serves as an auxiliary tool.

2.3 Physical Laws in Complex Domain

Incorporating with Eq.(2.2.7a), Eq.(2.2.13) can be recast into the form

$$\boldsymbol{p}\bullet\boldsymbol{p}+2m\left(\frac{\partial S}{\partial t}+V\right)=i\hbar\nabla\bullet\boldsymbol{p}\,. \tag{2.3.1}$$

This is the main result derived in (Baker-Jarvis and Kabos, 2003), indicating a new interpretation of Schrödinger equation in terms of complex energy conservation. When the external potential V is independent of time, corresponding to the case of a conservative complex dynamic system, solution of Schrödinger 's equation has the form

$$\Psi(\boldsymbol{q},t)=\psi(\boldsymbol{q})e^{-i(E/\hbar)t}\,, \tag{2.3.2}$$

and the associated complex action becomes

$$S(\boldsymbol{q},t)=\frac{\hbar}{i}\ln\Psi=\frac{\hbar}{i}\ln\psi(\boldsymbol{q})-Et\,. \tag{2.3.3}$$

The complex Hamilton-Jacobi equation (22b) then reads

$$H\left(\boldsymbol{q},\boldsymbol{p},t\right)=\frac{1}{2m}\,\boldsymbol{p}\bullet\boldsymbol{p}+V(\boldsymbol{q})+\frac{\hbar}{2mi}\nabla\bullet\boldsymbol{p}=-\frac{\partial S}{\partial t}=E\,. \tag{2.3.4}$$

This is the energy conservation law in complex mechanics, which manifests that the total energy in complex mechanics comprises three terms: kinetic energy $\boldsymbol{p}\bullet\boldsymbol{p}/(2m)$, external potential energy $V(\boldsymbol{q})$, and intrinsic potential energy $Q(\boldsymbol{q})=\hbar/(2mi)\nabla\bullet\boldsymbol{p}$. Conservative systems in complex mechanics are defined to be the systems that have constant total energy given by Eq.(2.3.4). Classical Hamiltonian $H(\boldsymbol{p},\boldsymbol{q})=\boldsymbol{p}\bullet\boldsymbol{p}/(2m)+V(\boldsymbol{q})$ therefore is not a constant in complex mechanics. We can say that the physical meaning underlying time-independent Schrödinger equation is just the energy conservation law in complex mechanics. This fact can be easily verified by substituting Eq.(2.2.7a) into the energy conservation law (2.3.4) with S given by Eq.(2.3.3), and the result turns out to be

$$\frac{\hbar^2}{2m}\nabla^2\psi(\boldsymbol{q})+\big(E-V(\boldsymbol{q})\big)\psi(\boldsymbol{q})=0\,, \tag{2.3.5}$$

which is just the time-independent Schrödinger 's equation.

Next we search for the counterpart of Newton's equation in complex mechanics. The first form of complex Newton's equation is found immediately from Eq.(2.2.6b):

$$\frac{dp_j}{dt}=-\frac{\partial H}{\partial q_j}=-\frac{\partial V}{\partial q_j}-\frac{\partial Q}{\partial q_j},\quad p_j,\ q_j\in\mathbb{C},\quad j=1,\ 2,\ 3\,. \tag{2.3.6}$$

This equation has the form of Newton's second law in the complex domain, in which the particle is subject to a complex quantum force $-\nabla Q$ in addition to the classical force $-\nabla V$. Using $p_j=m\dot{q}_j$ from Eq.(2.2.6a) and Q from Eq.(2.2.16), we can rewrite Eq.(2.3.6) in a more tractable form:

$$m\ddot{q}_j=\frac{\hbar^2}{2m}\frac{\partial}{\partial q_j}\big(\nabla^2\ln\Psi\big)-\frac{\partial V}{\partial q_j},\qquad j=1,\,2,\,3\,. \tag{2.3.7}$$

Apparently, complex motion is state or context dependent so that different state Ψ possesses different equation of motion. For a conservative system, further simplification can be made from Eq.(2.3.4) by noting that total potential is now in the form of

$$V_{\text{Total}}=Q+V=E-\frac{1}{2m}(\nabla S)^2=E+\frac{\hbar^2}{2m}(\nabla\ln\Psi)^2\,. \tag{2.3.8}$$

A succinct expression of complex Newton's equation then turns out to

$$m\ddot{q}_j = -\frac{\partial V_{\text{Total}}}{\partial q_j} = -\frac{\hbar^2}{2m}\frac{\partial}{\partial q_j}(\nabla \ln \Psi)^2, \quad j=1,\ 2,\ 3\,. \tag{2.3.9}$$

Since the equation of motion is independent of the constant E, we can choose $E=0$ as the reference energy level for V_{Total}. The magnitude of the total potential barrier now becomes

$$|V_{\text{Total}}| = \left|\frac{\hbar^2}{2m}(\nabla \ln \Psi)^2\right| = \frac{\hbar^2}{2m}\frac{|\nabla\Psi|^2}{\Psi^*\Psi}\,, \tag{2.3.10}$$

which states that the height of the total potential barrier is inversely proportional to $\Psi^*\Psi$. A spatial point with large value of $\Psi^*\Psi$ corresponds to the location with small potential barrier and hence large accessibility to this point. This fact legitimates the use of $\Psi^*\Psi$ as the probability measurement for a particle to appear at a specified spatial point.

Eq.(2.3.6) can be expressed alternatively in terms of the complex covariant derivative (Nottale, 1996) defined in Eq.(2.1.8). The left-hand side of Eq.(2.3.6) can be rewritten as

$$\frac{dp_j}{dt} = \frac{\partial p_j}{\partial t} + \frac{dq_1}{dt}\frac{\partial p_j}{\partial q_1} + \frac{dq_2}{dt}\frac{\partial p_j}{\partial q_2} + \frac{dq_3}{dt}\frac{\partial p_j}{\partial q_3} = \frac{\partial p_j}{\partial t} + \dot{\boldsymbol{q}}\bullet\nabla p_j\,, \tag{2.3.11}$$

where the complex velocity $\dot{\boldsymbol{q}} = \boldsymbol{p}/m$ can be expressed in terms of the wave function Ψ from Eq.(2.2.17) and Eq.(2.2.18) as

$$\dot{\boldsymbol{q}} = \frac{\boldsymbol{p}}{m} = \frac{\hbar}{mi}\nabla \ln \Psi\,. \tag{2.3.12}$$

Comparing Eq.(2.3.12) with Eq.(2.1.14), we have

$$\boldsymbol{\mathcal{V}} = \boldsymbol{v} - i\,\boldsymbol{u} = \dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_R + i\,\dot{\boldsymbol{q}}_I\,, \tag{2.3.13}$$

which implies that the complex velocity $\boldsymbol{\mathcal{V}}$ obtained from the Brown-motion velocities $\boldsymbol{v}$ and $\boldsymbol{u}$ is indeed the time derivative of the complex position $\boldsymbol{q}$. Further simplification of the second term in the right-hand side of Eq.(2.3.6) can be made by using the definition of $Q(\boldsymbol{q},t)$ in Eq.(2.2.4):

$$\begin{aligned}\frac{\partial Q(\boldsymbol{q},t)}{\partial q_j} &= \frac{1}{2m}\left(\frac{\hbar}{i}\right)\left(\frac{\partial^2 p_1}{\partial q_j \partial q_1} + \frac{\partial^2 p_2}{\partial q_j \partial q_2} + \frac{\partial^2 p_3}{\partial q_j \partial q_3}\right)\\ &= \frac{1}{2m}\left(\frac{\hbar}{i}\right)\left(\frac{\partial^3 S}{\partial q_j \partial q_1 \partial q_1} + \frac{\partial^3 S}{\partial q_j \partial q_2 \partial q_2} + \frac{\partial^3 S}{\partial q_j \partial q_3 \partial q_3}\right) \qquad ,\\ &= \frac{1}{2m}\left(\frac{\hbar}{i}\right)\left(\frac{\partial^2 p_j}{\partial q_1 \partial q_1} + \frac{\partial^2 p_j}{\partial q_2 \partial q_2} + \frac{\partial^2 p_j}{\partial q_3 \partial q_3}\right) = \frac{1}{2m}\left(\frac{\hbar}{i}\right)\nabla^2 p_j\end{aligned} \tag{2.3.14}$$

where the identity $p_i = \partial S/\partial q_i$ from Eq.(2.2.7a) has been used in the above derivation. Combining the results of Eq.(2.3.6), Eq.(2.3.11), and Eq.(2.3.14) yields

$$\frac{\partial p_j}{\partial t} + \boldsymbol{\mathcal{V}}\bullet\nabla p_j - i\frac{\hbar}{2m}\nabla^2 p_j = -\frac{\partial V}{\partial q_j}, \quad j=1,\ 2,\ 3\,. \tag{2.3.15}$$

In terms of the complex time derivative $d\!\!\!^{-}/dt$ defined in Eq.(2.1.8), the above equation is reduced to

$$\frac{d\!\!\!^{-}}{dt}p_j = -\frac{\partial V}{\partial q_j}\,. \tag{2.3.16}$$

This equation is regarded as the second form of complex Newton's equation, which was derived by Nottale (Nottale, 1996; Pissondes, 1998) by a different way. Nottale defined the new complex time derivative $d\!\!\!^{-}/dt$ in order to include the effect of quantum potential Q and made the complex Newton's equation (2.3.16) contain explicitly only the classical potential V; while in complex mechanics, conventional time derivative d/dt is employed but with quantum potential considered separately as shown in Eq.(2.3.16). Starting from complex mechanics, we thus have reproduced the main result of Nottale's fractal space-time approach to quantum mechanics. Since infinitely many geodesic lines between any couple of points is allowed in a fractal space-time, the equivalence between complex mechanics approach and fractal space-time approach implies that the multi-path phenomenon needs to be an intrinsic property of complex mechanics. In the next section we will identify this multi-path phenomenon as a projection effect from complex space to real space.

2.4 Multiple Path Behavior

In classical mechanics, particle trajectory is uniquely determined when an initial point $(\boldsymbol{q}(t_0), \boldsymbol{p}(t_0))$ in the phase space is given. This property is not valid for quantum mechanics, nor for complex mechanics. To investigate the non-uniqueness of the complex trajectory, which is governed by the complex Hamilton equation with initial conditions given in Eqs.(2.2.6), the easiest way is in terms of the solution of Schrödonger equation. For the convenience of explanation, we first consider a conservative system with separable potential V:

$$V(q) = V_1(q_1) + V_2(q_2) + V_3(q_3)\,, \tag{2.4.1}$$

More general case will be considered later. The solution of Schrödonger equation (2.3.5) with $V(q)$ given by Eq.(2.4.1) has a separable form:

$$\psi(q_1, q_2, q_3) = \psi_1(q_1)\psi_2(q_2)\psi_3(q_3) \tag{2.4.2}$$

with each ψ_j satisfying the following complex ordinary differential equations

$$\frac{\hbar^2}{2m}\frac{d^2\psi_j(q_j)}{dq_j^2} + (E_j - V_j(q_j))\psi_j(q_j) = 0, \quad j=1,\ 2,\ 3\,, \tag{2.4.3}$$

where $E_1 + E_2 + E_3 = E$ is the total energy of the system. Let ψ_{A_j} and ψ_{B_j} be two linearly independent solution of Eq.(2.4.3), then the general solution of Eq.(2.4.3) can be expressed as

$$\psi_j(q_j) = A_j\psi_{A_j}(q_j) + B_j\psi_{B_j}(q_j), \quad j=1,\ 2,\ 3 \tag{2.4.4}$$

It is worth noting that ψ_{A_j} and ψ_{B_j} depend on the value of E_j, which may or may not be the eigenvalue of Eq.(2.4.3). Hence, ψ_{A_j} and ψ_{B_j} are not restricted to be square-integrable functions, since they are not necessarily interpreted as probability density functions in complex mechanics. Employing Eq.(2.2.17) and Eq.(2.4.4), we obtain momentum p_j as

$$p_j(q_j) = \frac{\hbar}{i}\frac{\psi_j'(q_j)}{\psi_j(q_j)} = \frac{\hbar}{i}\frac{A_j\psi_{A_j}'(q_j) + B_j\psi_{B_j}'(q_j)}{A_j\psi_{A_j}(q_j) + B_j\psi_{B_j}(q_j)} = \frac{\hbar}{i}\frac{\psi_{A_j}'(q_j) + c_j\psi_{B_j}'(q_j)}{\psi_{A_j}(q_j) + c_j\psi_{B_j}(q_j)}\,, \tag{2.4.5}$$

where prime denotes the differentiation with respect to q_j. The coefficient $c_j = B_j / A_j$ is determined by applying the initial conditions given in Eq.(2.2.6b):

$$c_j = \frac{\left(ip_j^0 / \hbar\right)\psi_{A_j}(q_j^0) - \psi_{A_j}'(q_j^0)}{\psi_{B_j}'(q_j^0) - \left(ip_j^0 / \hbar\right)\psi_{B_j}(q_j^0)} \tag{2.4.6}$$

Thus momentum p_j as a function of q_j is uniquely determined by using this c_j in Eq.(2.4.5). The particle's trajectory $q_j(t)$ is then found from Eq.(2.2.18)

$$\frac{dq_j}{dt} = \frac{\hbar}{mi}\frac{\psi_{A_j}'(q_j) + c_j\psi_{B_j}'(q_j)}{\psi_{A_j}(q_j) + c_j\psi_{B_j}(q_j)}, \quad q_j(t_0) = q_j^0 \in \mathbb{C}, \quad j = 1,\ 2,\ 3 \tag{2.4.7}$$

which, after integration, gives q_j as an implicit function of time t

$$t = \frac{im}{\hbar}\int_{q_j^0}^{q_j}\frac{\psi_{A_j}(q_j) + c_j\psi_{B_j}(q_j)}{\psi_{A_j}'(q_j) + c_j\psi_{B_j}'(q_j)}\,dq_j \tag{2.4.8}$$

Notice that the integration in Eq.(2.4.8) is along complex contour, and as we are concerned with non-relativistic case here, the time t in Eq.(2.4.7) or (2.4.8) is real. In relativistic complex mechanics discussed later, time t is treated on a completely equal footing with spatial coordinates so that time is complexified in a similar way: $t = t_R + i\ t_I \in \mathbb{C}$.

Hence, for given initial conditions (q_i^0, p_i^0) and given energy E_i, a unique trajectory can be specified in complex space. However, the uniqueness of trajectory in complex space does not imply the uniqueness of trajectory in real space, since different points in complex space may be projected into the same point in real space. Initial point $(\boldsymbol{q}^0, \boldsymbol{p}^0)$ in complex phase space has real and imaginary

parts

$$\boldsymbol{q}^0 = \boldsymbol{q}_R^0 + i\,\boldsymbol{q}_I^0, \quad \boldsymbol{p}^0 = \boldsymbol{p}_R^0 + i\,\boldsymbol{p}_I^0. \tag{2.4.9}$$

If we fix the real-part initial condition $\boldsymbol{q}_R^0 = \overline{\boldsymbol{q}}_R^0$ and $\boldsymbol{p}_R^0 = \overline{\boldsymbol{p}}_R^0$, but let $\boldsymbol{q}_I^0$ and $\boldsymbol{p}_I^0$ be varying in $\mathbb{R}^3$, then a set of initial points in complex phase space is formed：

$$\Omega_0 = \left\{(\boldsymbol{q}^0, \boldsymbol{p}^0)\middle|\boldsymbol{q}_R^0 = \overline{\boldsymbol{q}}_R^0, \boldsymbol{p}_R^0 = \overline{\boldsymbol{p}}_R^0;\ \boldsymbol{q}_I^0, \boldsymbol{p}_I^0 \in \mathbb{R}^3\right\} \tag{2.4.10}$$

From this initial set, we can generate a set of complex trajectories

$$\boldsymbol{Q} = \left\{\boldsymbol{q}(t, \boldsymbol{q}^0, \boldsymbol{p}^0)\middle|(\boldsymbol{q}^0, \boldsymbol{p}^0) \in \Omega_0\right\} \tag{2.4.11}$$

by solving the complex Hamilton equations (2.2.6), with each trajectory $\boldsymbol{q}(t, \boldsymbol{q}^0, \boldsymbol{p}^0)$ starting from a different initial point $(\boldsymbol{q}^0, \boldsymbol{p}^0) \in \Omega_0$. The projection of $\boldsymbol{Q}$ into real space, however, gives rise to a set of real trajectories, as if they were all originated from the same initial point $(\overline{\boldsymbol{q}}_R^0, \overline{\boldsymbol{p}}_R^0)$ in real space. Thus, when we view from real space, we may observe an infinite number of trajectories connecting a couple of points, and this is just the multi-path phenomenon considered in Feynman's fractal space-time approach to quantum mechanics (Yang, 2005B). If we could view from complex space, this multi-path phenomenon disappears, since those trajectories connecting the two fixed points in real space no longer connect the same points in complex space by taking into account the different imaginary parts that the two connected points may have.

Aside from the varying imaginary parts of initial conditions, the other factor causing multi-path effect is the indeterminacy of the energy E_i in Eq.(2.4.3). From Eq.(2.3.4), E_i can be evaluated from the initial condition as

$$E_i = \frac{1}{2m}\left(p_i^0\right)^2 + V_i(q_i^0) + \frac{\hbar}{2mi}\left(\frac{dp_i}{dq_i}\right)_{q_i^0}, \quad i=1, 2, 3 \tag{2.4.12}$$

However, given the initial point $(\boldsymbol{q}^0, \boldsymbol{p}^0)$ in phase space we can only fix the first two terms in the right-hand side of Eq.(2.4.12). But to fully determine the energy E_i, an additional initial condition on $(dp_i / dq_i)_{\boldsymbol{q}^0}$ must be known. The dependency of the complex trajectory on the initial condition $(dp_i / dq_i)_{\boldsymbol{q}^0}$ yields a remarkable difference between complex mechanics and classical mechanics wherein particle's trajectory is uniquely determined by the initial point $(\boldsymbol{q}^0, \boldsymbol{p}^0)$. The dependency on this additional initial condition explains one of the reasons why fixing initial condition $(\boldsymbol{q}^0, \boldsymbol{p}^0)$ cannot fix particle's path in quantum mechanics. The other reason, as we have mentioned, is the indeterminacy of the imaginary part of the given initial conditions $(\boldsymbol{q}_I^0, \boldsymbol{p}_I^0)$.

There are special eigen-trajectories corresponding to stationary solution in quantum mechanics, along which energy E_i is precisely equal to the energy eigenvalues in Eq.(2.4.3) to ensure the existence of bound solution ($\psi_i \to 0$ as $|q_i| \to \infty$). For such eigen-trajectories, the random motion caused by multi-path effect is solely due to the indeterminacy of the imaginary parts of the given initial conditions. Accordingly, as we have known, stationary solution exhibits comparatively less randomness in comparison with the non-stationary solution.

The above discussion of multi-path effect is focused on conservative system with separable potential. For a general system, we have to solve the time-dependent Schrödonger equation (2.2.15) directly. By noting that Eq.(2.2.15) has second-order derivative in $\boldsymbol{q}$ and first-order derivative in t, a general solution for Eq.(2.2.15) must contain seven free constants in the form of

$$\Psi = \Psi(t, \boldsymbol{q}, \boldsymbol{c}) = \Psi(t, q_1, q_2, q_3, c_1, \cdots, c_7) \tag{2.4.13}$$

In terms of this general solution, the constant $\boldsymbol{c}$ can be found from the initial condition $(\boldsymbol{q}^0, \boldsymbol{p}^0)$ using the relations in Eq.(2.2.7a):

$$\boldsymbol{p}^0 = \nabla S\big|_{\boldsymbol{q}=\boldsymbol{q}^0} = \frac{\hbar}{i}\frac{\nabla\Psi(t_0, \boldsymbol{q}^0, \boldsymbol{c})}{\Psi(t_0, \boldsymbol{q}^0, \boldsymbol{c})} \tag{2.4.14}$$

There are three equalities in Eq.(2.4.14), which in general are not enough to fully determine the seven components in the constant vector $\boldsymbol{c}$. Let $\boldsymbol{C}_0$ be the set consisting all the constant vectors $\boldsymbol{c}$ satisfying Eq.(2.4.14). Particle's trajectory $\boldsymbol{q}(t)$ is obtained from Eq.(2.2.18):

$$\frac{dq_i}{dt} = \frac{\hbar}{mi}\frac{\partial\Psi(t,\boldsymbol{q},\boldsymbol{c})/\partial q_i}{\Psi(t,\boldsymbol{q},\boldsymbol{c})}, \quad q_i(t_0) = q_i^0, \quad i\text{=}1,\ 2,\ 3 \tag{2.4.15}$$

For each $\boldsymbol{c}' \in \boldsymbol{C}_0$, there is a corresponding trajectory $\Gamma_{\boldsymbol{c}'}(t)$ determined by Eq.(2.4.15) using $\boldsymbol{c} = \boldsymbol{c}'$. Then the set $\{\Gamma_{\boldsymbol{c}}(t) : \boldsymbol{c} \in \boldsymbol{C}_0\}$ comprises a class of complex trajectories all originating from the same initial point $(\boldsymbol{q}^0, \boldsymbol{p}^0)$. That is to say, complex trajectory computed from the complex Hamilton equations is not fixed by giving an initial point $(\boldsymbol{q}^0, \boldsymbol{p}^0)$ in complex phase space. Even for the special case of conservative system with separable potential as mentioned earlier, where complex trajectory can be uniquely determined, real trajectory still cannot be fixed by giving $(\boldsymbol{q}_R^0, \boldsymbol{p}_R^0)$, since all complex initial points $(\boldsymbol{q}_R^0 + i\,\boldsymbol{q}_I^0, \boldsymbol{p}_R^0 + i\,\boldsymbol{p}_I^0)$ with fixed real part $(\boldsymbol{q}_R^0, \boldsymbol{p}_R^0)$ and varying imaginary part $(\boldsymbol{q}_I^0, \boldsymbol{p}_I^0)$ are projected into the same point $(\boldsymbol{q}_R^0, \boldsymbol{p}_R^0)$ in real space.

In short, the non-uniqueness of complex trajectories and their projections into real space naturally induce the multi-path phenomenon observed in real space, which, in turn, produces the wave behavior of a particle. From an ontological viewpoint of complex mechanics, what objectively exists is particle's motion in complex space, while the wave motion associated with a particle is merely a phenomenon of projection of particle's complex motion into real space. This ontological view is different from that of de Broglie-Bohm approach where both particle and wave objectively exist so that the wave motion, which is governed by the Schrödinger equation, guides the motion of particle.

2.5 Comparison with Bohmian Mechanics

In the above discussion we start from the complex Hamilton equations and arrive at the Schrödonger equation. What will be happened if we reverse the process ? This problem has been partially studied by Bohm (1952). Based on Madelung's hydrodynamic formulation of quantum mechanics, and using de Broglie's concept of pilot waves, Bohm developed a formalism in which the initial and final states of a process are connected causally by quantum trajectories. de Broglie-Bohm theory has been proved as a useful tool to study quantum motion (Holland, 1993). Many successful applications of de Broglie-Bohm theory were reported in the literature, such as definition of arrival times (Leavens, 1998), quantum chaos (Parmenter and Valentine, 1995), photo-dissociation (Dey, Askar and Rabitz, 1998), and simulation of resonant tunneling diodes (Oriols, Martin, and Sune, 1996), etc. de Broglie-Bohm theory deals with real quantum trajectory, while the complex mechanics considered here deals with complex quantum trajectory; the latter may be conceived of as a complex extension of the former. Since real space is a subset of complex space, all the properties of real quantum trajectory are retained in complex space, and thus all the existing merits of de Broglie-Bohm formalism are still reserved in complex mechanics; however, many properties of complex quantum trajectory are lost, when projected into real space. Consequently, there are quantum behaviors, which are explicable in terms of complex mechanics, cannot be explained using de Broglie-Bohm theory.

Bohm's approach begins with rewriting wave function Ψ in so-called Madelung form (de Broglie ansatz)

$$\Psi = R_B\, \mathrm{e}^{iS_B/\hbar}, \quad R_B,\ S_B \in \mathbb{R}\,, \tag{2.5.1}$$

and is followed by inserting Ψ into the Schrödonger 's equation (2.2.15) and separating into real and imaginary parts to yield two coupled nonlinear partial differential equations:

$$\frac{\partial S_B}{\partial t} + \frac{(\nabla S_B)^2}{2m} - \frac{\hbar^2}{2m}\frac{\nabla^2 R_B}{R_B} + V = 0 \tag{2.5.2}$$

$$\frac{\partial R_B^2}{\partial t} + \nabla\bullet\left(\frac{R_B^2 \nabla S_B}{m}\right) = 0 \tag{2.5.3}$$

Eq.(2.5.2) has the form of the classical Hamilton-Jacobi equation apart from the extra term

$$Q_B = -\frac{\hbar^2}{2m}\frac{\nabla^2 R_B}{R_B} = -\frac{\hbar^2}{2m}\frac{\nabla^2 |\Psi|}{|\Psi|}, \tag{2.5.4}$$

which is known as the (real) quantum potential. The subscript B is introduced to highlight the

results obtained from Bohm's approach. Quantum potential Q_B is real, since it is constructed from the real function $R_B = |\Psi|$; while the quantum potential Q defined in Eq.(2.2.4) is complex, and if expressed in terms of Ψ, it takes the form

$$Q = -\frac{\hbar^2}{2m}\left[\frac{\nabla^2\Psi}{\Psi} - \frac{(\nabla\Psi)^2}{\Psi^2}\right], \tag{2.5.5}$$

which contains an additional term $(\nabla\Psi/\Psi)^2$ in comparison with Q_B. To ensure that Born's probability interpretation of Ψ is explicable from the viewpoint of classical statistical mechanics, Bohm made a special assumption about the momentum of a particle:

$$m\dot{\boldsymbol{q}}_B = \boldsymbol{p}_B = \nabla S_B, \tag{2.5.6}$$

where S_B is the phase of Ψ as defined in Eq.(2.5.1). It is noted that S_B is real and the trajectory $\boldsymbol{q}_B(t)$ obtained from Eq.(2.5.6) is, of course, real. As a comparison, S in Eq.(2.2.14) is complex, and the trajectory obtained from Eq.(2.2.7a) is also complex. Moreover, Eq.(2.5.6) is a postulate of Bohm's approach, while Eq.(2.2.7a) is not a postulate of complex mechanics, but is derived from the Hamilton equation (2.2.6).

Applying the operator ∇ to Eq.(2.5.2) and employing the relation in Eq.(2.5.6), we have

$$\frac{d}{dt}(m\dot{\boldsymbol{q}}_B) = -\nabla(V + Q_B) \tag{2.5.7}$$

In comparison with the complex Newton's equation (2.3.6), Eq.(2.5.7) describes particle's motion in real space with real deriving force $-\nabla(V + Q_B)$. It is emphasized that the real trajectory computed from Eq.(2.5.6) or (2.5.7) is not the one obtained from projecting Eq.(2.3.6) into real space, since the latter has included the interaction effect between the real and imaginary parts of the complex quantum potential in Eq.(2.2.4), while the former only exhibits the effect induced from the real part of Eq.(2.2.4). Trajectory obtained from Eq.(2.5.6) or Eq.(2.5.7) therefore only reflects partial information of quantum phenomena and it is not surprising that many efforts in the literature have been made to modify the guidance condition (2.5.6) to widen its validity.

One counterintuitive result of Bohm's original theory is the prediction that particles in any eigenstate described by

$$\Psi(\boldsymbol{q}, t) = \psi_n(\boldsymbol{q})e^{-iE_n/\hbar} \tag{2.5.8}$$

are motionless, since $\boldsymbol{p}_B = \nabla S_B = \nabla(-E_n/\hbar) = 0$ from Eq.(2.5.6). To remedy this defect, Bohm and Vigier (1954) suggested that Eq.(2.5.6) and (2.5.7) only represent some regular motion, while the actual motion has an additional irregular fluctuating component, which is effectively random. Inspired by the analogy with real fluids, they assumed that the velocity of an individual particle is given by

$$\dot{\boldsymbol{q}}_B = \dot{\bar{\boldsymbol{q}}}_B + \xi(t) = \frac{\nabla S_B}{m} + \xi(t) \tag{2.5.9}$$

where $\xi(t)$ represents a chaotic contribution to the velocity of that particle, which fluctuates in a way that may be represented as random but with zero average. This means that we should regard $\nabla S_B/m$ appeared in equation (2.5.6) as the average velocity $\dot{\bar{\boldsymbol{q}}}_B$ rather than its actual velocity. The effect of random velocity $\xi(t)$ is automatically accounted for in the framework of complex mechanics. It can be shown that what Eq.(2.5.6) predicts is just the equilibrium point of some nonlinear complex dynamics. Indeed, according to complex mechanics the complex trajectory for a particle described by the wave function (2.5.8) can be found from Eq.(2.2.18)

$$\frac{dq_j}{dt} = -i\frac{\hbar}{m}\frac{\partial\psi_n(\boldsymbol{q})/\partial q_j}{\psi_n(\boldsymbol{q})}, \quad q_j(0) = q_{R_j}(0) + i\ q_{I_j}(0) \in \mathbb{C} \tag{2.5.10}$$

The integration of this equation gives the characteristic eigen-trajectories belonging to the eigen-state (2.5.8). For a given $\boldsymbol{q}_R(0)$, the eigen-trajectory is not uniquely determined, since $\boldsymbol{q}_I(0)$ may be varying and cannot be controlled from real world. The projection of the ensemble of trajectories with fixed $\boldsymbol{q}_R(0)$ and varying $\boldsymbol{q}_I(0)$ into real space constitutes the associated wave motion of the particle.

What Bohm's original theory predicts is the situation when the particle locates at the equilibrium points where $\partial\psi_n(\boldsymbol{q})/\partial q_j = 0$ and consequently, $d\boldsymbol{q}/dt = 0$ from Eq.(2.5.10) leads to the conclusion of a motionless particle.

The other correction of Eq.(2.5.6) comes from the consideration of spin effect. Holland (Holland, 1999, 2003) pointed out that Bohm's guidance condition (2.5.6) is relevant only to spin-0 particles. For particles with spin, this equation is inconsistent if the theory is to be ultimately embedded in a relativistic theory. The condition of Lorentz covariance on the law of motion implies that the momentum of a particle with spin $\boldsymbol{s}$ must be given by

$$\boldsymbol{p}_B = \nabla S_B + (\nabla \log \rho)\times \boldsymbol{s}, \tag{2.5.11}$$

where $\rho = R_B^2 = |\Psi|^2$. As for complex mechanics, it will be shown later that the spin motion is an intrinsic property of the complex momentum given in Eq.(2.2.17) and the additional term $(\nabla \log \rho)\times \boldsymbol{s}$ in Eq.(2.5.11) is not necessary in Eq.(2.2.7a).

For a given initial condition $(\boldsymbol{q}_B^0, \boldsymbol{p}_B^0)$, Eq.(2.5.6) uniquely determines the real trajectory $\boldsymbol{q}_B(t)$, just like the case of classical mechanics. Consequently, multi-path behavior does not come out naturally from Bohm's approach. To conquer this difficulty, Bohm proposed that although in theory, trajectory $\boldsymbol{q}_B(t)$ is uniquely determined by the initial condition $(\boldsymbol{q}_B^0, \boldsymbol{p}_B^0)$, in practice $\boldsymbol{q}_B(t)$ is not fixed since practical considerations inevitably limit the extent of our knowledge of $(\boldsymbol{q}_B^0, \boldsymbol{p}_B^0)$. Hence, Bohm suggested that we have to consider an ensemble of systems possessing a range of initial conditions consistent with our imperfection knowledge of the actual system. If the statistical distribution $P(\boldsymbol{q}_B^0, t_0)$ of these admissible initial points satisfies

$$P(\boldsymbol{q}_B^0, t_0) = \left|\Psi(\boldsymbol{q}_B^0, t_0)\right|^2 \tag{2.5.12}$$

then the guidance condition (2.5.6) and the conservation of probability flow in Eq.(2.5.3) ensure the compatibility of the Bohm's predictions with those of standard quantum mechanics, i.e., $P(\boldsymbol{q}_B, t) = \left|\Psi(\boldsymbol{q}_B, t)\right|^2$, $\forall t \geq t_0$. In other words, Bohm attributes the multi-path behavior of quantum system to our imperfection knowledge of initial conditions, while complex mechanics identifies the multi-path behavior as an intrinsic property of complex motion, which is independent of our knowledge of initial conditions; even if (real part) initial conditions are precisely known, particle's multi-path behavior still exists.

To show that assumption (2.5.12) is attainable, Bohm and Vigier (1954) determined the properties of the random motions described by $\xi(t)$ in Eq.(2.5.9) so that an arbitrary initial ensemble with a probability distribution $P(\boldsymbol{q}_B^0, t_0)$ will approach an ensemble with a probability distribution $P(\boldsymbol{q}_B, t) = |\psi(\boldsymbol{q}_B, t)|^2$ after a suitable interval of time. An explicit characterization of the random velocity $\xi(t)$ was derived later by Bohm and Hiley (1989):

$$\dot{\boldsymbol{q}}_B = \dot{\bar{\boldsymbol{q}}}_B + \xi(t) = \frac{\nabla S_B}{m} + \mathcal{D}\frac{\nabla R_B}{R_B}, \tag{2.5.13}$$

where S_B and R_B are defined in Eq.(2.5.1), and $\mathcal{D}$ is a diffusion coefficient. This random velocity $\xi(t)$ is actually the osmotic velocity originated from the Brownian motion. From the viewpoint of complex mechanics, it is enlightening to find that the inclusion of the random velocity $\xi(t)$ in Eq.(2.5.13) is just to take into account the influence of the imaginary velocity $\dot{\boldsymbol{q}}_I(t)$ on the real velocity $\dot{\boldsymbol{q}}_R(t)$. This fact can be seen by relating the velocity $\dot{\boldsymbol{q}}_B$ in Eq. (2.5.13) to the complex velocity defined in Eq. (2.2.18). Applying the wave function defined in Eq.(2.5.1) to Eq.(2.2.18), we have

$$\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_R + i\,\dot{\boldsymbol{q}}_I = \frac{\boldsymbol{p}}{m} = \frac{\nabla S}{m} = \frac{\hbar}{mi}\frac{\nabla\Psi}{\Psi}\bigg|_{\Psi = R_B e^{iS_B/\hbar}} = \frac{\nabla S_B}{m} - i\frac{\hbar}{m}\frac{\nabla R_B}{R_B} \tag{2.5.14}$$

If the diffusion coefficient in Eq.(2.5.13) is chosen as $\mathcal{D} = \hbar/m$, then the comparison between Eq.(2.5.13) and Eq.(2.5.14) gives

$$\dot{\boldsymbol{q}}_R = \frac{\nabla S_B}{m} = \dot{\bar{\boldsymbol{q}}}_B, \quad \dot{\boldsymbol{q}}_I = -\frac{\hbar}{m}\frac{\nabla R_B}{R_B} = -\xi\,. \tag{2.5.15}$$

This result reveals that the real part of the complex velocity is equal to the mean velocity of the particle as observed from the real space and the imaginary part of the complex velocity, although cannot be observed directly in real word, has measurable influence on the mean velocity $\dot{\bar{\boldsymbol{q}}}_B$ in terms of Brown motion, which induces an extra velocity component $\xi(t)$ to it. In other words, we can say that what is known by Brown motion is the motion induced by the interaction between the real and imaginary parts of the particle's complex motion.

In the above discussion we have mentioned several attempts to modify the guidance condition (2.5.6). Indeed, further investigation (Holland, 1999, 2003) points out that the guidance condition (2.5.6) is not the only one that produces the probability distribution consistent with standard quantum mechanics. Any guidance law in the form of

$$\boldsymbol{p}_B = \nabla S_B + \frac{\boldsymbol{a}}{R_B^2}, \quad \nabla \bullet \boldsymbol{a} = 0 \ , \tag{2.5.16}$$

still maintains empirical compatibility with quantum mechanics. This fact indicates a problem of indeterminacy in the particle's velocity and causes a tension in Bohmian mechanics (Cushing, 1996). There are two different physical views, i.e., "the causal view" and "the guidance view", which coexist uneasily in Bohm's theory. The tension that we find in Bohm's theory is the contingency of the initial velocity $\dot{\boldsymbol{q}}_B(0)$ in the causal view governed by the Newtonian interpretation in Eq.(2.5.7) and its non-contingency in the guidance view governed by Eq.(2.5.6) or Eq.(2.5.16). In causal view, the initial velocity $\dot{\boldsymbol{q}}_B(0)$ is a contingency in the same sense as the initial position of the particle – neither is implicit in the equation of motion (2.5.7). The contingency of $\dot{\boldsymbol{q}}_B(0)$ indicates that $\dot{\boldsymbol{q}}_B(0)$ may not happen to be the one assigned by the guidance condition (2.5.6) or (2.5.16). Hence, $\dot{\boldsymbol{q}}_B = \nabla S_B / m$ may not hold in the initial time and the Born's probability interpretation is not always recoverable from classical statistics.

The inconsistency between Eq.(2.5.6) and Eq.(2.5.7) stems primarily from the fact that the initial conditions used in solving Eq.(2.5.7) is not employed in the determination of S_B in Eq.(2.5.6). Such a tension or inconsistency does not appear in complex mechanics. As we have addressed in section 1.4, the same set of initial conditions has been used in solving the Schrödonger equation (2.2.15) and the complex guidance law (2.2.7a), as well as in solving the Newton's equation (2.3.6), which gives the complex causal view. Therefore, causal view and guidance view are consistent in complex mechanics.

Another deterministic approach, which also claims to resolve the aforementioned difficulties of Bohm's approach, is the Floyd-Faraggi-Matone (FFM) trajectory representation (Floyd, 1982, 1894; Faraggi and Matone, 1999). Instead of using the guidance law (2.5.6), they employed the Jacobi's equation $t - t_0 = \partial S_B / \partial E$ as the equation of motion. Similar approach was proposed by Bouda and Djama (2001) but with a modified S_B. The FFM trajectory, like dBB trajectory, is in real domain and does not claim equivalence with standard quantum mechanics in the predictions of all observed phenomena.

2.6 Complex Variables with Memory

As we have seen in the previous sections, particles moving in complex space have complex positions $(x, y, z) \in \mathbb{C}$ and complex momenta $(p_x, p_y, p_z) \in \mathbb{C}$ and hence require complex coordinates to describe their motion. When we face a particle's complex trajectory, some useful tools coming from complex variable theory will greatly facilitate the analysis and simplify the solution. Complex variables treated in mathematics are assumed to be memoryless; they have no memory about their past motions. On the contrary, complex variables in complex Hamilton mechanics have a memory of their past trajectories recorded by the Hamilton equations of motion. Upon applying complex variable theory to a dynamic system operating in a complex domain, we shall notice the fundamental changes brought out by a complex variable with memory. In this section we first introduce the concept of complex variables with memory (with dynamics) and then discuss their impact on the definitions of analytic functions, multi-valued functions, and contour integrations. We point out the necessary remedies for the conventional complex variable theory to cope with complex variables with memory. Understanding these remedies is crucial to the establishment of the

complex-extended Hamilton mechanics in the next chapter and elsewhere in this book.

2.6.1 Solving Complex Dynamics Equations

Strictly speaking, in complex mechanics we often encounter functions of several complex variables. For example, the Hamiltonian $H(x,y,z)$ describing motion in 3D complex space is a function of three complex variables x, y, and z. For dynamic systems having separable solutions, theory of single complex variable is enough; while for dynamic systems having inseparable solutions, all the physical quantities need be considered as functions of three complex variables x, y, and z, simultaneously. In this case, formal theory of several complex variables is required.

Mathematically, the complex variables x, y, and z in a function $F(x,y,z)$ are conventionally regarded as independent variables; they can be varied independently and assigned arbitrarily, as though they were memoryless and without dynamics. But in complex mechanics, complex coordinates x, y, and z are not independent variables; they describe the motion of a particle and are considered to be functions of time t. When evaluating a function $F(x,y,z)$, we first need to specify the equations of motion for the complex variables x, y, and z in the form of

$$\frac{dx}{dt} = f_1(x,y,z), \quad \frac{dy}{dt} = f_2(x,y,z), \quad \frac{dz}{dt} = f_3(x,y,z), \quad t \geq 0\,. \tag{2.6.1}$$

The complex variable x, y, and z are said to have memories, because they carry a time record for their trajectory $(x(t),y(t),z(t))$ which is determined from Eq.(2.6.1) with a given initial point $(x(0),y(0),z(0))$. We can evaluate the function $F(x(t),y(t),z(t))$ along the trajectory at any specified time t. By contrast, in the mathematical theory of complex variable we usually treat x, y, and z as memoryless complex variables such that they can be arbitrarily assigned in the function $F(x,y,z)$.

For relativistic motions, time t and spatial coordinates x, y, and z are all complex-valued functions of the proper time τ, and the equations of motion are otherwise given by

$$\frac{dt}{d\tau} = f_1(t,x,y,z), \ \frac{dx}{d\tau} = f_2(t,x,y,z), \ \frac{dy}{d\tau} = f_3(t,x,y,z), \ \frac{dz}{d\tau} = f_4(t,x,y,z), \ \tau \geq 0\,, \tag{2.6.2}$$

A solution of the above set of equations $(t(\tau),x(\tau),y(\tau),z(\tau))$ describes a particle's trajectory (also called world line) in the 4D spacetime as the proper time τ evolves. Again we see that the coordinates τ, x, y, and z are complex variables carrying memories about their past motions.

On solving Eq.(2.6.1) numerically, we have to decompose x, y, z, f_1, f_2, and f_3 into real and imaginary parts:

$$x = x_R + ix_I, \quad y = y_R + iy_I, \quad z = z_R + iz_I \tag{2.6.3a}$$

$$f_1 = f_{1R} + if_{1I}, \quad f_2 = f_{2R} + if_{2I}, \quad f_3 = f_{3R} + if_{3I} \tag{2.6.3b}$$

This composition transforms the three complex ODEs in Eq.(2.6.1) into the following six real ODEs:

$$\frac{dx_R}{dt} = f_{1R}(x_R,x_I,y_R,y_I,z_R,z_I), \quad \frac{dy_R}{dt} = f_{2R}(x_R,x_I,y_R,y_I,z_R,z_I), \quad \frac{dz_R}{dt} = f_{3R}(x_R,x_I,y_R,y_I,z_R,z_I)$$
$$\frac{dx_I}{dt} = f_{1I}(x_R,x_I,y_R,y_I,z_R,z_I), \quad \frac{dy_I}{dt} = f_{2I}(x_R,x_I,y_R,y_I,z_R,z_I), \quad \frac{dz_I}{dt} = f_{3I}(x_R,x_I,y_R,y_I,z_R,z_I)$$

for which numerical integration can be performed in a usual way. It is noted that the decomposition involved in Eq.(2.6.3b) is taken numerically at some instant t; decomposition in functional form is not necessary.

Example 2.6.1

Consider the following ODE involving only one complex variable:

$$\frac{dx}{dt} = i\tan x, \quad x(0) = 0\,. \tag{2.6.4}$$

It can be seen that no solution exists if x is restricted to the real axis. We next search in the complex

domain for a possible solution. Three methods may be available for this problem. The first approach is to apply the decomposition $x = x_R + ix_I$ to separate Eq.(2.6.4) into real part and imaginary part:

$$\frac{dx_R}{dt} = -\frac{\sinh(x_I)\cosh(x_I)}{\cos^2(x_R)+\sinh^2(x_I)}, \quad \frac{dx_I}{dt} = \frac{\sin(x_R)\cos(x_R)}{\cos^2(x_R)+\sinh^2(x_I)}. \tag{2.6.5}$$

and then invoke a numerical scheme to integrate the above coupled ODEs with initial condition $x_R(0) = x_I(0) = 0$. The decomposition of Eq.(2.6.4) into Eq.(2.6.5) is performed analytically. In many cases analytical decomposition is often unavailable, and numerical decomposition is used instead:

$$\frac{dx_R}{dt} = \mathrm{Re}\big(i\tan(x(t))\big), \quad \frac{dx_I}{dt} = \mathrm{Im}\big(i\tan(x(t))\big), \quad x_R(0) = x_I(0) = 0\,, \tag{2.6.7}$$

where $\mathrm{Re}(i\tan(x(t)))$ and $\mathrm{Im}(i\tan(x(t)))$ are evaluated numerically at some discrete instant t_k, $k = 0,\ 1,\ \cdots,\ n$. In some special cases, analytical integration is possible and the present problem happens to be this case. Because $\tan x$ is an analytic function of x, its integration can be performed as in the real case:

$$x(t) = \arcsin(e^{it})\,. \tag{2.6.8}$$

Most of the quantum trajectories considered in the later chapters are found numerically by the same method employed in Eq.(2.6.7)

2.6.2 Analytic Functions

A function $f(z)$ that is single-valued and differentiable at all points of a domain R is said to be analytic in R. A function $f(z)$ is differentiable at the point z, if the derivative

$$f'(z) = \lim_{\Delta z\to 0}\frac{f(z+\Delta z) - F(z)}{\Delta z}\,. \tag{2.6.9}$$

exists and is unique. In other words, the differentiability at z requires the value of $f'(z)$ defined in Eq.(2.6.9) independent of the direction from which Δz tends to zero. This requirement leads us to the well- known Cauchy-Riemann condition (refer to problem 2.1 for its derivation):

$$\frac{\partial f_R}{\partial x_R} = \frac{\partial f_I}{\partial x_I} \quad \text{and} \quad \frac{\partial f_I}{\partial x_R} = -\frac{\partial f_R}{\partial x_I}, \tag{2.6.10}$$

where we have employed the decomposition used in Eqs.(2.6.3): $z = z_R + iz_I$ and $f(z) = f_R(z_R, z_I)$ $+if_I(z_R, z_I)$.

Example 2.6.2

Check whether the following two functions are analytic: (1) $f(z) = z_R^2 - z_I^2 + 2iz_Rz_I$, (2) $f(z) = 2z_I + iz_R$.

Answer:

(1) $f_R = z_R^2 - z_I^2$, $f_I = 2z_Rz_I$:

$$\frac{\partial f_R}{\partial z_R} = 2z_R = \frac{\partial f_I}{\partial z_I} \quad \text{and} \quad \frac{\partial f_I}{\partial z_R} = 2z_I = -\frac{\partial f_R}{\partial z_I}.$$

(2) $f_R = 2z_I$, $f_I = z_R$:

$$\frac{\partial f_R}{\partial z_R} = 0 = \frac{\partial f_I}{\partial z_I} \quad \text{and} \quad \frac{\partial f_I}{\partial z_R} = 1 \neq -2 = -\frac{\partial f_R}{\partial z_I}.$$

Therefore, the function $f(z) = z_R^2 - z_I^2 + 2iz_Rz_I$ is analytic, but $f(z) = 2z_I + iz_R$ is not.

□

It can be further shown by using the Cauchy-Riemann condition that if $f(z)$ is analytic then $f(z)$ can not be a function of z^*. Hence, any expression representing an analytic function of z can contain z_R and z_I in the combination $z_R + iz_I$, not in the combination $z_R - iz_I$. Because the expression $f(z) = z_R^2 - z_I^2 + 2iz_Rz_I = (z_R + iz_I)^2 = z^2$ does not contain z^* explicitly, it is analytic. But as to the function $f(z) = 2z_I + iz_R = -i(z - 3z^*)/2$, it contain z^* explicitly, and thus is not an

analytic function.

The requirement that an analytic function $f(z)$ must have a unique derivative along all the possible directions of z is understandable, because when z is treated as an independent, it may change in arbitrary direction. However, in complex Hamilton mechanics, the direction along which the derivative of $f(z)$ is taken is not arbitrary, because z here is a complex variable with memory and thus the change of z must be continuous with respect to its past trajectory, i.e., must satisfy the dynamical constraints, such as those in Eq.(2.6.1). That is to say, what is concerned in a dynamical system is not the omnidirectional derivative $df(z)/dz$ defined in Eq.(2.6.9), but is the unidirectional derivative $df(z(t))/dt$ along a trajectory $z(t)$ determined from the equations of motion.

Example 2.6.3

Consider the complex function $f(z) = 2z_I + iz_R = -i(z - 3z^*)/2$, which is not analytic as verified in example 2.6.6. In this example we wish to evaluate the directional derivative of $f(z)$ along a trajectory satisfying

$$\frac{dz}{dt} = iz, \quad z(0) = 1. \tag{2.6.11}$$

The trajectory satisfying the prescribed initial condition can be solved readily as $z(t) = e^{it} = \cos t + i \sin t$, and the function $f(z)$ evaluated on this trajectory becomes

$$f(z(t)) = 2z_I + iz_R = 2\sin t + i\cos t,$$

which is differentiable at any $t \geq 0$ in spite of its being not analytic. We can see from this example that equations of motion such as Eq.(2.6.11) restrict the derivative of $f(z)$ to be evaluated along some prescribed direction and the demand for the analyticity of $f(z)$ becomes unnecessary.

□

2.6.3 Multi-Valued Function and Branch Cuts

Many complex functions have multiple values. Functions, such as $\sqrt{z}$ and $\ln z$, have different values for the same value of z. In complex variable theory, we usually select suitable branch cut to restrict the argument of z such that a given complex function $f(z)$ becomes single-valued. Shall we need a branch cut to evaluate $f(z)$ uniquely if z is the complex coordinate of a particle moving under the guidance of some prescribed equations of motion? Our answer is negative, because the complex variable z here has a memory about its past trajectory by which the ambiguity of the argument of z can be removed. To elucidate this point, we first review the occurrence of multi-valued functions and recall how to fix it by setting a branch cut. Then we will see that the past memory of a complex variable can be used to remove the multiplicity of a multi-valued function.

Let us consider the multi-valued function $f(z) = z^{1/2}$ and express z as $z = re^{i\theta}$. Referring to Fig.2.6.1a, it is evident that, as the point z traverses any closed contour C that does not enclose the origin, θ will return to its original value after one complete circuit. However, for any closed contour C' that does enclose the origin, after one circuit, θ increases to $\theta + 2\pi$ (see Fig.2.6.1b). Hence, for the function $f(z) = z^{1/2}$, after one circuit

$$r^{1/2}\exp(i\theta/2) \to r^{1/2}\exp\left[(i(\theta + 2\pi)/2)\right] = -r^{1/2}\exp(i\theta/2). \tag{2.6.12}$$

In other words, the value of $f(z)$ changes around any closed loop enclosing the origin; in the present case $f(z) \to -f(z)$. Thus $z = 0$ is a branch point of the function $f(z) = z^{1/2}$. In order to treat $f(z)$ as single-values, the conventional approach is to define a branch cut in the Argand diagram. A branch cut is a line or curve in the complex plane and may be regarded as an artificial barrier that we must not cross. Branch cuts are positioned in such a way that we are prevented from making a complete circuit around any branch point, and so the function in question remains single-valued. For the function $f(z) = z^{1/2}$, we may take as a branch cut any curve starting at the origin and extending to $|z| = \infty$ in any direction, since all such curves would equally well prevent us from making a closed loop around the branch point at the origin. It is usual, however, to take the cut along either the real or the imaginary axis. For example, in Fig.2.6.1c, we take the cut as the positive real axis. By agreeing

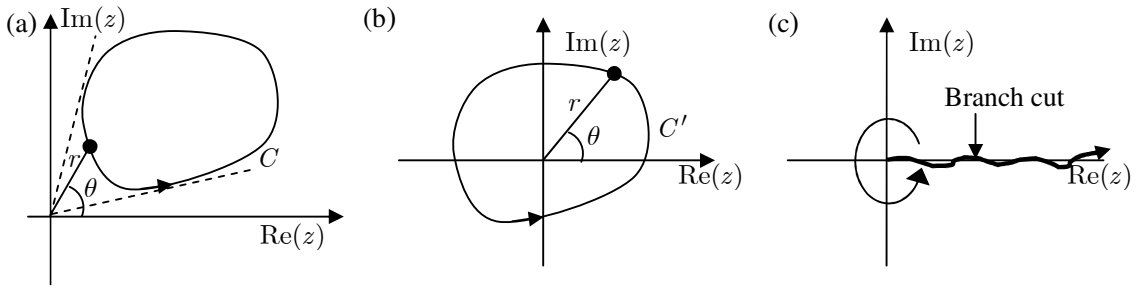

Fig.2.6.1 (a) A closed contour not enclosing the origin; (b) a closed contour enclosing the origin; (c) a possible branch cut for $f(z)=\sqrt{z}$ limits the argument of z in the range $0\le\theta<2\pi$.

not to cross this cut, we restrict θ to lie in the range $0\le\theta<2\pi$, and so keep $f(z)$ single-valued.

The other way to make a function single-valued is to record the motion of the point z in terms of the time t. Shown in Fig.2.6.2a is a circle which has no memory about how many times a particle has passed the point P. It is this memoryless property that produces the ambiguity of the argument of z at P:

$$z=r\exp\left[i(\pi/4+2n\pi)\right],\quad n\in\mathbb{Z}. \tag{2.6.13}$$

Once we evaluate $z^{1/2}$ or $\ln z$ with z given above, we then obtain the multi-value results for different values of n. From this point of view, the existence of a multi-valued function is actually based on the tacit assumption that the complex point z is memoryless about its past trajectory. On the contrary, if z has memory about its past trajectory, we can count the times it has passed the point P and uniquely determine the integer n in Eq.(2.6.13), making multi-value effect no longer exist.

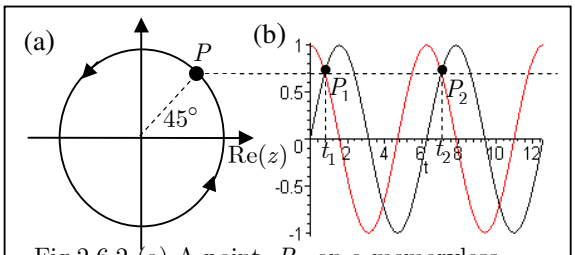

Fig.2.6.2 (a) A point P_1 on a memoryless circle; (b) A memory record for a particle moving on the circle.

In Fig.2.6.2b, we have a time record for the trajectory of the complex point z:

$$z(t)=\cos t+i\sin t=e^{it}. \tag{2.6.14}$$

When $t=t_1=\pi/4$, it is the first time that the moving point z passes the fixed point P, and when $t=t_2=\pi/4+2\pi$, it is the second time that z passes P. In this way, we see that $z(t)$ expressed by Eq.(2.6.14) has a memory about its past trajectory. Taking the square root of $z(t)$ then yields

$$z^{1/2}=e^{it/2}=\cos(t/2)+i\sin(t/2),\quad t\in\mathbb{R}, \tag{2.6.15}$$

which indicates that there is only one value for $z^{1/2}$ at any instant t. That is to say, the function $f(z)=z^{1/2}$ becomes single-valued if z has a record or a memory of its past motion.

In complex Hamilton mechanics, a time record for z is always available from the equations of motion, which enables us to remove the multiplicity of any function involving z. The conventional approach of setting a branch cut on z to yield a single-valued function can not be applied to a complex variable z with memory, because branch cut may induce an artificial movement of z, which may be consistent with the intrinsic dynamics of z. Let us manifest this point by the following example.

Example 2.6.4

In this example we intend to find the value of $\ln z$ with the motion of z described by Eq.(2.6.11). If we neglect the dynamics inherent in z, z becomes memoryless and the value of $\ln z$ turns out to be the conventional one:

$$\ln z = \ln\big[r\exp(i\theta)\big] = \ln\big[r\exp(i(\theta+2n\pi))\big] = \ln r + i(\theta+2n\pi), \quad n\in\mathbb{Z}. \tag{2.6.16}$$

It appears that $\ln z$ is an infinitely multi-valued function of a memoryless complex variable z. The principle value of $\ln z$ is obtained by taking $n=0$ so that its argument lies in the range $-\pi<\theta\le\pi$:

$$\ln z = \ln r + i\theta, \quad -\pi<\theta\le\pi, \tag{1.6.18}$$

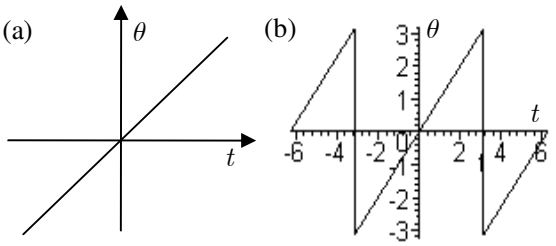

Fig.2.6.3 (a) The correct argument of z is proportional to time t. (b) The principle value of $\ln z$ limits the argument in the range $-\pi<\theta\le\pi$.

which amounts to setting a branch cut on the negative real axis. However, z has its own dynamics expressed by $z(t)=e^{it}=\cos t+i\sin t$, which shows that the point z is moving counterclockwise on the unit circle. The correct argument θ of z is equal to the time t and is monotonically increasing as depicted in Fig.2.6.3a. But the argument θ found from the principle value of $\ln z$ according to Eq.(1.6.18) is limited to the range $-\pi<\theta\le\pi$ and forms a zigzag pattern as shown in Fig.2.6.3b, where it can be seen that once the argument θ is beyond π, it is forced to reset to $-\pi$. This discontinuity on θ is caused by the branch cut that prevents the point z from crossing the negative real axis. But this artificial prevention violates the intrinsic dynamics of z, a constant rotation motion along the unit circle, and significantly distorts the correct argument as shown in Fig.2.6.3a.

□

This example tells us that a complex variable z having intrinsic dynamics or memory can automatically remove the multiplicity of a multi-valued function without introducing any branch cut. The insertion of a superfluous branch cut may induce an inconsistent restriction on the motion of z, which violates the intrinsic dynamics of z.

2.6.4 Contour Integration and Residue Theorem

Consider the integration of a complex function $f(z)$ along a closed contour c:

$$I=\oint_c f(z)dz, \tag{2.6.19}$$

where we assume that $f(z)$ is continuous within and on the contour c and is analytic, except for a finite number of poles, within c. Then residue theorem states that the contour integration I is equal to $2\pi i$ times the sum of the residues of $f(z)$ at its poles within c, i.e.,

$$I=\oint_c f(z)dz = 2\pi i\sum_k R_k, \tag{2.6.20}$$

where R_k is the residue of $f(z)$ evaluated at its k^{th} pole. Suppose the k^{th} pole locates at $z=z_k$ with order m. Then the residue R_k is given by the formula:

$$R_k=\lim_{z\to z_k}\left\{\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\big[(z-z_k)^m f(z)\big]\right\}. \tag{2.6.21}$$

The residue theorem stated in Eq.(1.6.20) is developed for a memoryless complex variable z so that the contour c can be arbitrarily chosen and the integration around c can be either clockwise or counterclockwise (referring to Fig.2.6.4, the convention is counterclockwise). However, this freedom of arbitrarily choosing the integration contour and the integration direction is not applicable to a complex variable with memory. For a complex variable with intrinsic dynamics, the contour c must be one of the closed trajectories that is actually traced out by the moving point z, and the direction around the contour c must be determined by the equations of motion for z.

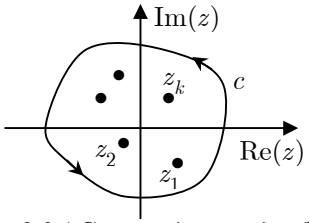

Fig.2.6.4 Contour integration for a memoryless complex variable can freely choose the shape and direction of the contour c.

Consider a complex variable with the following intrinsic dynamic

$$\frac{dz}{dt} = F(z)\,, \tag{2.6.22}$$

and suppose c_k is the closed contour generated from the above intrinsic dynamics with the initial position $z(0) = z_0^{(k)}$. The contour c_k may be conceived of as the following point set:

$$c_k \triangleq \left\{ z^{(k)}(t) \middle| \, t \geq 0 \right\} = \left\{ z(t) \middle| dz \,/\, dt = F(z),\ z(0) = z_0^{(k)},\ t \geq 0 \right\}. \tag{2.6.23}$$

Given every point $z^{(k)}(t)$ on the contour c_k, the contour integration I in Eq.(2.6.19) along the contour c_k can now be written as

$$I_k = \oint_{c_k} f(z)dz = \int_0^T f(z^{(k)}(t))F(z^{(k)}(t))dt\,, \tag{2.6.24}$$

where T is the time to complete one loop of c_k. Eq.(2.6.24) is the desired expression of contour integration for a complex variable z with intrinsic dynamics. Therein we can see that the contour c_k is not chosen artificially but is determined from the dynamics (2.6.22) with the initial point $z_0^{(k)}$. Although the direction of integrating dz around c_k is not explicitly specified in Eq.(2.6.24), it is otherwise determined automatically by Eq.(2.6.22). Therefore, both the shape and direction of the contour are self-determined by the intrinsic dynamics of the complex variable z in question, which are the two main features different from the contour integration for memoryless complex variables.

Accordingly, when we apply the residue theorem (2.6.20) to the contour integration (2.6.24) for a complex variable z with memory, we shall first make sure of two things: (1) whether the contour c is one of the closed trajectories traced out by z, and (2) which is the direction of integration around c, clockwise or counterclockwise?

Example 2.6.5

This problem is to evaluate the following contour integration

$$\oint_c \frac{dz}{z(z-2)}, \quad c: \ |z| = 1\,, \tag{2.6.25}$$

for a complex variable z with intrinsic dynamics

$$\frac{dz}{dt} = iz\,. \tag{2.6.26}$$

The contour c along which the integration is taken is the unit circle. So we need to check whether the unit circle is one of the trajectories produced by Eq.(2.6.26). The solution of Eq.(2.6.26) is given readily as

$$z(t) = Ce^{it}, \quad z(0) = C\,, \tag{2.6.27}$$

which can be in the form of a unit circle $|z| = 1$ as long as $|C| = 1$, i.e., if the initial position $z(0)$ falls on the unit circle. Hence, the demanded contour integration in Eq.(2.6.25) is admissible under the dynamic constraint (2.6.26). Let us take $z(0) = 1$ and rewrite Eq.(2.6.25) with the substitution $z(t) = e^{it}$ as

$$\oint_c \frac{dz}{z(z-2)} = \int_0^{2\pi} \frac{idt}{(e^{it}-2)} = i\int_0^{2\pi} \frac{dt}{(\cos t - 2) + i\sin t}\,, \tag{2.6.28}$$

where the upper limit $T = 2\pi$ is the time to complete one loop of the unit circle. In the above expression we have transformed the complex integral in (2.6.25) to a real time integral. Separating Eq.(2.6.28) into real and imaginary parts, we obtain the final result of contour integration:

$$\oint_c \frac{dz}{z(z-2)} = \int_0^{2\pi} \frac{\sin t}{(\cos t - 2)^2 + \sin^2 t}dt + i\int_0^{2\pi} \frac{\cos t - 2}{(\cos t - 2)^2 + \sin^2 t}dt = -i\pi\,. \tag{2.6.29}$$

Next we evaluate Eq.(2.6.25) by using residue theorem. The solution $z(t) = e^{it}$ from Eq.(2.6.26) indicates that the point z is moving counterclockwise on the unit circle and is thus in the same contour direction assumed in Eq.(2.6.20). Because there is only one pole $z = 0$ within the unit

circle, the contour integration is then equal to $2\pi i$ times the residue at $z=0$:

$$\oint_c \frac{dz}{z(z-2)} = 2\pi i \,\mathrm{Res}\big|_{z=0} = 2\pi i \frac{1}{0-2} = -i\pi \,, \tag{2.6.30}$$

which is identical to Eq.(2.6.29) obtained by time integration.

For the purpose of comparison, we consider the counterpart of Eq.(2.6.26):

$$\frac{dz}{dt} = -iz \,, \tag{2.6.31}$$

whose solution $z(t)=e^{-it}$ also represents a unit-circle contour but with clockwise direction, which is opposite to that assumed in Eq.(2.6.20).A minus sign must be included in Eq.(2.6.20) to cope with this opposite contour direction, leading to the result:

$$\oint_c \frac{dz}{z(z-2)} = -2\pi i \,\mathrm{Res}\big|_{z=0} = -2\pi i \frac{1}{0-2} = i\pi \,. \tag{2.6.32}$$

Special attention is paid to the distinction in the symbols of integral between Eq.(2.6.30) and Eq.(2.6.32). The integration directions in these two integrals are not specified by convention or by artificial selection; instead, they are self-determined by the intrinsic dynamics of the complex variable z given, respectively, by Eq.(2.6.26) and Eq.(2.6.31). Without the information of intrinsic dynamics, we do not know which of Eq.(2.6.30) and Eq.(2.6.32) is to be applied.

On the other hand, if the time integral in Eq.(2.6.24) is used instead of the complex integral, we need not bother about the direction of the contour. Substituting Eq.(2.6.31) and $z(t)=e^{-it}$ into Eq.(2.6.25), we obtain the correct result directly

$$\oint_c \frac{dz}{z(z-2)} = -i\int_0^{2\pi} \frac{dt}{(\cos t-2)+i\sin t} = i\pi \,. \tag{2.6.33}$$

It is observed that an additional minus sign appears automatically in Eq.(2.6.33), which is contributed from the minus sign in the relation $dz=-izdt$ in Eq.(2.6.31).

2.7 Chapter Summary

As we have seen, complex mechanics introduced in this chapter is nothing but the Hamilton mechanics embedded in complex space. By simply extending the domain of classical mechanics to complex space, this chapter reveals the possibility of describing quantum motion in terms of classical concepts and tools. The fundamental principles of complex mechanics derived above are all based on two postulates: (1) all physical quantities are complex valued, and (2) every particle experiences an intrinsic complex quantum potential Q in addition to the applied potential V. With these two postulates, the predictions of complex mechanics are found to be consistent with the existing results:

(1) The derived complex Hamilton equations are equivalent to those obtained from Nelson's Brownian approach, and from Nottale's and Elnaschie's fractal space-time approach to quantum mechanics.

(2) The non-unique complex trajectory obtained from the solution of the complex Hamilton equation produces the expected multiple paths, as proposed by Feynman. the observed wave behavior of a particle is shown to be stemmed from the multi-path phenomenon caused by the non-uniqueness of particle's complex trajectories and their projections into real space.

(3) The derived complex Hamilton-Jacobi equation is found to be identical to Schrödinger equation. From the viewpoint of complex mechanics, Schrödinger equation with its complex nature is a statement of energy conservation law in complex space and its relation to the total potential $V+Q$ justifies the use of its solution ψ as a probability measure for a particle to appear at a specific spatial point.

Particle's coordinates in complex mechanics are complex variables with memory; they possess intrinsic dynamics such that their behavior is self-determined and can not be specified artificially. We point out in this chapter the necessary remedies for the conventional complex variable theory to cope with complex variables with memory.

2.8 Problems

2.1 In complex mechanics, coordinate (x, y, z) and momentum (p_x, p_y, p_z) are all treated as complex variables, and a wavefunction $\psi(x, y, z)$ or an action function $S(x, y, z)$ is treated as function of complex variables. When we perform the gradient operation on S to obtain ∇S, such as in Eq.(2.2.7a), we tacitly assume that $S(x, y, z)$ is an analytic function so that its derivatives with respect to x, y, and z exist. A complex function $f(z)$ is said to be analytic in a domain R, if it is single-valued and differentiable at all points in R. The differentiability of $f(z)$ means that its derivative

$$f'(z) = \lim_{\Delta z \to 0} \left[\frac{f(z + \Delta z) - f(z)}{\Delta z} \right] \tag{2.7.1}$$

exists and is unique, i.e., its value does not depend upon the direction from which Δz approaches to zero. If we let $f(z) = u(x, y) + iv(x, y)$ and $\Delta z = \Delta x + i\Delta y$, then we have

$$f(z + \Delta z) = u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y), \tag{2.7.2}$$

and the limit in Eq.(4a) is given by

$$\lim_{\Delta x, \Delta y \to 0} \left[\frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x, y) - iv(x, y)}{\Delta x + i\Delta y} \right]. \tag{2.7.3}$$

(a) By assuming $\Delta z \to 0$ along the real axis, i.e., $\Delta z = \Delta x$ and $\Delta y = 0$, evaluate the limit in Eq.(2.7.3).

(b) By assuming $\Delta z \to 0$ along the imaginary axis, i.e., $\Delta z = i\Delta y$ and $\Delta x = 0$, evaluate again the limit in Eq.(2.7.3).

(c) The differentiability of $f(z)$ requires that the results obtained in (a) and (b) must be equal. Show that the required condition is just the Cauchy-Riemann relation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{2.7.4}$$

(d) Since x and y are related to z and its complex conjugate z^* by

$$x = \frac{1}{2}(z + z^*), \quad y = \frac{1}{2i}(z - z^*), \tag{2.7.5}$$

we may formally regard any function $f = u + iv$ as a function of z and z^*, rather than x and y. If we do this and examine $\partial f / \partial z^*$, we obtain

$$\frac{\partial f}{\partial z^*} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z^*}. \tag{2.7.6}$$

Show that if f is analytic, then f cannot be a function of z^*, i.e., $\partial f / \partial z^* = 0$. Hint: Apply Eq.(2.7.5) to Eq.(2.7.6) and use the Cauchy-Riemann condition (2.7.4).

(e) The result of (d) implies that an analytic function of z can contain x and y only in the combination $x + iy$, not in the combination $x - iy$. From this point of view, show that $f(z) = x^2 - y^2 + i2xy$ is differentiable, while $f(z) = 2y + ix$ is not differentiable.

2.2 In the definition of an analytic function, one of the conditions imposed was that the function is singled-valued. A multi-valued function can still be treated as analytic if its principle value is taken. In polar form, a complex number z can be written as

$$z = re^{i\theta} = re^{i(\theta + 2n\pi)}, \quad n \in \mathbb{Z}, \quad -\pi < \theta \leq \pi \tag{2.7.7}$$

Taking the logarithm of both sides, we find

$$\mathrm{Ln}(z) = \ln(r) + i(\theta + 2n\pi). \tag{2.7.8}$$

Hence, the logarithm function $\mathrm{Ln}(z)$ is multi-valued, depending on the value of n. The principle value of $\mathrm{Ln}(z)$ is denoted by $\ln(z)$, which is evaluated at $n = 0$, i.e.,

$$\ln(z) = \ln(r) + i\theta, \quad -\pi < \theta \le \pi\,. \tag{2.7.9}$$

According to the above definition, find the all possible values of $\mathrm{Ln}(-i)$ and its principle value.

2.3 As can be seen in Eq.(2.7.7), the argument of a complex number is multi-valued, i.e., $\arg(z) = \theta + 2n\pi$. However, in complex mechanics, the complex number z is not arbitrary; it may be a position coordinate of a particle whose motion is governed by some prescribed equations, such as Eq.(2.2.18). For example, consider the following equation of motion for z:

$$\frac{dz}{dt} = iz, \quad z(0) = 1\,. \tag{2.7.10}$$

Show that it has the solution

$$z(t) = e^{it}, \quad t \ge 0\,. \tag{2.7.11}$$

It can be seen that the argument of z is uniquely determined by time t and no arbitrariness is allowed as in Eq.(2.7.7). A complex function becomes multi-valued in the presence of the arbitrariness of the argument of z. For instance, $f(z) = z^{1/6}$ is multi-values, because at the same z point in the complex plane

$$z = r\exp\big[i(\theta + 2k\pi)\big], \quad k = 0,\ 1,\ \cdots,\ 5 \tag{2.7.12}$$

$f(z) = z^{1/6}$ has six possible values:

$$f(z) = r^{1/6}\exp\left[i\frac{(\theta + 2k\pi)}{6}\right], \quad k = 0,\ 1,\ \cdots,\ 5\,. \tag{2.7.13}$$

Therefore, to make $f(z)$ single-valued, we need restrict the range of $\arg(z)$. Now, you are asked to evaluate $f(z) = z^{1/6}$ with z given by Eq.(2.7.11). Can you have a multi-valued result at any instant of time? And why?

2.4 In order that $f(z)$ may be treated as single-valued, we may define a branch cut in the complex plane. Branch cuts are positioned in such a way that we are prevented from making a complete circuit around any one branch point, and so the function in question remains single-valued. Find the branch points of $f(z) = \sqrt{z^2 + 1}$, and sketch suitable arrangements of branch cuts. Also plot $f(z)$ as a function of time in the complex plane with z given by Eq.(2.7.11) and check whether you need to construct a branch cut to ensure an unique value of $f(z)$.